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CALCULATION OF THE MAXIMUM FORCE BETWEEN TWO COAXIAL CIRCULAR CURRENTS

By Frederick W. Grover

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I. INTRODUCTION

Aside from its mathematical interest, the problem of the calculation of the force which exists between two circular currents has received a good deal of attention, owing to its practical importance in the calculation of the constants of apparatus used in absolute measurements. In such practical cases, it is true, the current is not concentrated in a circular filament, but flows in a number of wires, generally so disposed as to form a circular coil of rectangular cross section. The calculation of the mutual inductance of two such coils, as well as the force exerted between them, is, however, most readily based on the ideal case, a correction being applied to take into account the difference between the actual distribution of the current and the ideal distribution. This correction may be made with all the precision necessary, even in the most refined work, by the methods of Rayleigh and Lyle, provided that the dimensions of the cross sections are small with respect to the radii of the coils and the distance between them. In the formula of quadratures¹ (Rayleigh) the coil is replaced by a number of circular filaments, the weighted mean being taken of their separate effects; in the method of Lyle² the

¹ Cray, *Absolute Meas.*, Vol. II, Part II, p. 403. This Bulletin, 2, p. 370; 1906 (Scientific Papers No. 42).

² Phil. Mag., 3, p. 310; 1902. Also, this Bulletin, 2, p. 374; 1906 (Scientific Papers No. 42).

coil is shown to be equivalent to two circular filaments, whose radii and positions depend on the dimensions of the cross section. For details of such calculations the reader is referred to the original articles and to articles in this Bulletin.³

The mutual energy of two currents is equal to the product of their mutual inductance by the two current strengths. Accordingly, the force exerted by one circular filament on the other, in any specified direction, is equal to the rate of change of the mutual inductance as the filament moves in that direction multiplied by the product of the two current strengths. The force between two circular currents of unit strength may, therefore, be found by differentiating any suitable formula for their mutual inductance. The formulas of Maxwell and Nagaoka derived in this way for the special case of parallel coaxial circles are given for convenience of reference in the following section. These formulas allow of the calculation of the force with an accuracy which is limited only by the precision attainable in the measurement of the dimensions of the coils and the calculation of the correction for the finite cross section.

The present paper has to do with the special case, very important in practice, where the two parallel, coaxial, circular filaments are placed with such a distance between their planes that their mutual attraction or repulsion is a maximum. This is the condition which is realized in the form of current balance, first used by Lord Rayleigh⁴ in 1884, and by Rosa, Dorsey, and Miller⁵ at the Bureau of Standards. In this form of balance a coil is suspended from one arm of a balance, with its plane midway between those of two horizontal fixed coils, the distances between the planes of the coils being so chosen that the forces exerted by the currents in the fixed coils on the current in the suspended coil are a maximum. This arrangement has the advantage that not only is the force sensibly constant for small displacements of the suspended coil from this ideal position, but the constant of the balance is a function of the *ratio* of the radii of the fixed and

³ This Bulletin, 2, pp. 397-399, 1906 (Scientific Papers No. 42); and 8, p. 323, 331, 1911 (Scientific Papers No. 169).

⁴ Phil. Trans., 175, pp. 411-460, 1884; Rayleigh's Scientific Papers, 2, pp. 278-332.

⁵ This Bulletin, 8, pp. 269-393; 1911 (Scientific Papers No. 169).

movable coils, which ratio can be found with great accuracy by electrical methods, thus rendering unnecessary the measurement of the individual radii. The position of the coils for maximum force can be found by experiment.

The value of the maximum force can be calculated from the known ratio of the radii by interpolation between a number of values of the force, calculated by known formulas, for various distances in the neighborhood of the experimentally determined correct position of the coils. As far as is known to the author, this is the only method which has been previously used. The formulas and methods here developed are presented as affording a more direct solution of the problem, and although the amount of numerical work is, in the general case, still appreciable a check is afforded on the methods previously employed. For such cases as occur in practice, the constants may be obtained with very little labor from the tables of fundamental values given below.

The main formulas (33) to (37) were derived by the author in June, 1911. Since then, however, other work has interfered to delay the development of the auxiliary formulas and the preparation of the tables.

II. FORMULAS FOR THE FORCE BETWEEN TWO COAXIAL CIRCULAR COILS.

1. MAXWELL'S FORMULA IN ZONAL HARMONICS.⁶

Maxwell expressed the mutual inductance between two coaxial circular currents, whose radii A and a subtend, respectively, the angles α_1 and α_2 at the origin, and whose planes lie at distances B and b from this point, by a formula in zonal harmonics

$$M = 4 \pi^2 c \sin^2 \alpha_1 \sin^2 \alpha_2 \left\{ \frac{1}{2} \frac{c}{C} P_1'(\alpha_1) P_1'(\alpha_2) + \dots \dots \dots \right. \quad (1)$$

$$\left. + \frac{1}{n(n+1)} \frac{c^n}{C^n} P_n'(\alpha_1) P_n'(\alpha_2) + \dots \dots \dots \right\}$$

where $c = \sqrt{a^2 + b^2}$ and $C = \sqrt{A^2 + B^2}$

Substituting for the zonal harmonics their well-known values in terms of sines and cosines, and introducing for the latter their

⁶ *Elect. and Mag.*, Vol. II, sec. 696.

values in terms of the radii and the distances b , B , and c , C , the formula (1) becomes

$$\begin{aligned}
 M = & 1.2 \pi^2 \frac{A^2 a^2}{C^3} + 2.3 \pi^2 \frac{A^2 B a^2 b}{C^5} \\
 & + 3.4 \pi^2 \frac{A^2 a^2 (B^2 - \frac{1}{4} A^2) (b^2 - \frac{1}{4} a^2)}{C^7} + 4.5 \pi^2 \frac{B b (B^2 - \frac{3}{4} A^2) (b^2 - \frac{3}{4} a^2) A^2 a^2}{C^9} \\
 & + 5.6 \pi^2 \frac{(B^4 - \frac{3}{2} B^2 A^2 + \frac{1}{8} A^4) (b^4 - \frac{3}{2} b^2 a^2 + \frac{1}{8} a^4) A^2 a^2}{C^{11}} \\
 & + 6.7 \pi^2 \frac{B b (B^4 - \frac{5}{2} B^2 A^2 + \frac{5}{8} A^4) (b^4 - \frac{5}{2} b^2 a^2 + \frac{5}{8} a^4) A^2 a^2}{C^{13}} + \dots
 \end{aligned} \quad (2)$$

which on differentiation with respect to b gives for the force⁷

$$\begin{aligned}
 F = \frac{\pi^2 A^2 a^2}{C^4} \left\{ 1.2.3 \frac{B}{C} + 2.3.4 \frac{b (B^2 - \frac{1}{4} A^2)}{C^3} + 3.4.5 \frac{B (B^2 - \frac{3}{4} A^2) (b^2 - \frac{1}{4} a^2)}{C^5} \right. \\
 + 4.5.6 \frac{B (B^4 - \frac{3}{2} B^2 A^2 + \frac{1}{8} A^4) (b^2 - \frac{3}{4} a^2) b}{C^7} \\
 \left. + 5.6.7 \frac{B (B^4 - \frac{5}{2} B^2 A^2 + \frac{5}{8} A^4) (b^4 - \frac{5}{2} b^2 a^2 + \frac{1}{8} a^4)}{C^9} + \dots \right\} \quad (3)
 \end{aligned}$$

On account of its slow convergence, the formula (3) is not suitable for numerical calculations. It is, however, valuable, in that it gives a clear view of the way the force varies when the distance between the coils is increased.

2. MAXWELL'S FORMULA IN ELLIPTIC INTEGRALS.⁸

An accurate expression for the force was derived by Maxwell by differentiating his well-known formula for the mutual inductance of two coaxial circles.

$$M = -4 \pi \sqrt{Aa} \left\{ \left(k - \frac{2}{k} \right) K + \frac{2}{k} E \right\}$$

the result being

$$F = - \frac{\pi z \sin \gamma}{\sqrt{Aa}} \{ 2 K - (1 + \sec^2 \gamma) E \} \quad (4)$$

where A and a are the radii of the two circles, z is the distance between their planes, and K and E are the complete elliptic integrals to modulus $k = \sin \gamma =$

$$\frac{2 \sqrt{Aa}}{\sqrt{(A+a)^2 + z^2}}$$

⁷ This Bulletin, 8, p. 325, 1911 (Scientific Papers No. 171).

⁸ Elect. and Mag., Vol. II, sec. 701.

A table was prepared by Lord Rayleigh giving the value of $\log [\sin \gamma \{2K - (1 + \sec^2 \gamma)E\}]$ to seven places of decimals.⁹ This table was recalculated at the Bureau of Standards, using Legendre's tables of elliptic integrals, the logarithms being carried out to ten places, and some small errors in the seventh place corrected. The table thus corrected was reproduced in the article on the absolute determination of the international ampere, by Rosa, Dorsey, and Miller.

3. NAGAOKA'S FORMULAS

The force between two parallel coaxial circles has also been expressed by Nagaoka¹⁰ in terms of the q series of Jacobi. These expressions have the advantage of rapid convergence and of not requiring the use of tables of elliptic integrals. Nagaoka's formula (29), extended by the author of the present paper to include four additional terms, is

$$F = \frac{192 \pi^2 z}{\sqrt{Aa}} q^{\frac{3}{2}} \left[1 + 20 q^2 + 225 q^4 + 1840 q^6 + 12120 q^8 + 68052 q^{10} + 337465 q^{12} + 1513740 q^{14} + 6247665 q^{16} + \dots \right] \quad (5)$$

Extending Nagaoka's formula (21), the author obtained the further formula which is equivalent to the formula (8) found by Nagaoka in a more recent article.¹¹

$$F = \frac{\pi z}{16 q_1 \sqrt{Aa}} \left[(1 + 12 q_1 - 192 q_1^2 + 1232 q_1^3 - 5634 q_1^4 + 21648 q_1^5 - 73600 q_1^6 + 226944 q_1^7 - 648189 q_1^8 + \dots) - 12 q_1 \log_e \frac{1}{q_1} \cdot (1 - 10 q_1 + 60 q_1^2 - 300 q_1^3 + 1300 q_1^4 - 4884 q_1^5 + 16320 q_1^6 - 49920 q_1^7 + 142500 q_1^8 - \dots) \right] \quad (6)$$

⁹ Phil. Trans., 175, pp. 411-460, 1884; Rayleigh's Scientific Papers, 2, 327.

¹⁰ Jour. College of Science, Tokyo, vol. 16, art. 15, p. 12, 1903; Phil. Mag., 6, p. 19, 1903.

¹¹ Tokyo Math.-Phys. Soc., vol. 6, p. 156, 1911.

where

$$\left. \begin{aligned} q &= \frac{l}{2} + 2 \left(\frac{l}{2} \right)^5 + 15 \left(\frac{l}{2} \right)^9 + \dots \\ \frac{l}{2} &= \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{k^2}{2(1+k')(1+\sqrt{k'})^2} \\ k &= \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + z^2}} & k' &= \sqrt{1 - k^2} = \frac{\sqrt{(A-a)^2 + z^2}}{\sqrt{(A+a)^2 + z^2}} \\ q_1 &= \frac{l_1}{2} + 2 \left(\frac{l_1}{2} \right)^5 + 15 \left(\frac{l_1}{2} \right)^9 + \dots \\ \frac{l_1}{2} &= \frac{1}{2} \frac{1 - \sqrt{k}}{1 + \sqrt{k}} = \frac{k'^2}{2(1+k)(1+\sqrt{k})^2} \end{aligned} \right\} \quad (6a)$$

In most cases only two or three terms need be calculated to give the force with a good deal of precision. The formulas are given at length here so as to give all needed precision, even in the most unfavorable cases. Formula (6) converges most rapidly for circles which are near together. It is not so convenient to use as (5) and will be employed only in those cases where the convergence of (5) is not satisfactory. With the number of terms included here, the force may be calculated by one or the other of the two formulas with an accuracy of better than 1 part in 100 000, even in the most unfavorable case. To facilitate calculations, Nagaoka has published¹² tables giving the values of $\frac{F\sqrt{Aa}}{z}$ for different values of q .

Further formulas for the force have also been given by Nagaoka, for which the original articles already cited should be consulted.

III. THE MAXIMUM FORCE

The foregoing formulas suffice to give the value of the force between the circles with all the precision required in the most careful work.

For the calculation of the constant of the Rayleigh current balance we need, as has already been explained, to obtain either

¹² Tokyo Math.-Phys. Soc., vol. 6, p. 154, 1911.

(a) an expression giving the maximum value of the force between the circular currents as a function of the ratio of their radii, or
 (b) a formula for calculating the value of the distance z_m which exists between their planes when the force is a maximum. The maximum force F_m may, in the latter case, be calculated by substituting z_m for z in any one of the formulas for the force given above, which converges satisfactorily.

Although more direct, the first method is, in some cases, less accurate than the second, and as a knowledge of the distance z_m is of interest in itself, formulas for both methods have been given below.

1. RAYLEIGH'S APPROXIMATE FORMULA FOR z_m

Obviously z_m may be found by differentiating any suitable expression for the force F with respect to the distance z between the circles, equating the result to zero, and solving for z . The result is the value z_m for which the force is a maximum. Provided the formula for z_m is not too complicated, it gives on substitution in the formula for the force an expression suitable for calculating the maximum force F_m .

The differentiation of Maxwell's formula (3) gives, when the origin is taken at the center of the smaller circle ($b=0$)

$$\left(\frac{dF}{dz}\right)_{b=0} = \frac{\pi^2 A^2 a^2}{C^4} \left[2 \cdot 3 \cdot 4 \frac{(z_m^2 - \frac{1}{4}A^2)}{C^3} - 3 \cdot 5 \cdot 6 \frac{a^2(z_m^4 - \frac{3}{2}z_m^2 A^2 + \frac{1}{8}A^4)}{C^7} + \dots \right] = 0 \quad (7)$$

This may be solved for z_m by a method of approximation. The first term shows that $\frac{z_m}{A}$ is less than unity. Taking as a first approximation that value of $\frac{z_m}{A}$ which makes the first term zero, and substituting this value in the second term, we find as a second approximation Rayleigh's equation ¹³

$$\frac{z_m^2}{A^2} = \frac{1}{4} \left(1 - \frac{9a^2}{5A^2} \right) \quad (8)$$

or

$$\frac{z_m}{A} = \frac{1}{2} \left(1 - \frac{9a^2}{10A^2} - \dots \right)$$

¹³ Rayleigh: Scientific Papers, vol. 2, p. 284.

It is not difficult, by substituting this value in the second and third terms of (7) to obtain a further approximation

$$\frac{z_m}{A} = \frac{1}{2} \left(1 - \frac{9a^2}{10A^2} - \frac{1}{4} \frac{a^4}{A^4} \right) \quad (9)$$

Unfortunately, the slow convergence of these expressions, and the rapid increase in the amount of labor necessary for obtaining each new term, prevent the use of (8) and (9) except for purposes of orientation, unless within the limited range where $\frac{a}{A}$ is small.

The differentiation of Maxwell's equation (4) in elliptic integrals is complicated and does not give a satisfactory formula for z_m .

By following the method of Nagaoka the author of the present paper has obtained expressions in the form of q series, which enable the values of z_m and F_m to be calculated with all the accuracy necessary in refined work, and these formulas have been tested thoroughly as to their applicability for numerical calculations and as to their agreement with the results found by other methods.

IV. DERIVATION OF THE NEW FORMULAS

The mutual inductance of two parallel coaxial circles of radii A and a , the distance between their planes being z , is given by the integral ¹⁴

$$M = 4\pi Aa \int_0^\pi \frac{\cos \theta \, d\theta}{\sqrt{A^2 + a^2 + z^2 - 2Aa \cos \theta}} \quad (10)$$

where θ is the angle between two elements ds and ds' of the two circles.

Differentiating this with respect to z , we find for the force

$$F = \frac{dM}{dz} = 4\pi Aa z \int_0^\pi \frac{\cos \theta \, d\theta}{(A^2 + a^2 + z^2 - 2Aa \cos \theta)^{\frac{3}{2}}} \quad (11)$$

and the condition which must hold when the force between the circles is a maximum is

$$\begin{aligned} \frac{dF}{dz} = \frac{d^2M}{dz^2} &= 4\pi Aa \int_0^\pi \frac{\cos \theta \, d\theta}{(A^2 + a^2 + z^2 - 2Aa \cos \theta)^{\frac{5}{2}}} \\ &+ 4\pi Aa z^2 \int_0^\pi \frac{3 \cos \theta \, d\theta}{(A^2 + a^2 + z^2 - 2Aa \cos \theta)^{\frac{5}{2}}} = 0 \end{aligned} \quad (12)$$

¹⁴ Maxwell, *Elect. & Mag.*; Vol. II, sec. 701.

Nagaoka in the first of the papers above cited has shown how to express the elliptic integral (11) in elliptic functions, and by expanding these in terms of q functions has obtained the formulas (5) and (6) (the latter in a different form) for the force, which have been given above.

For the integration of (12) we need evaluate only one new integral—the second. For completeness, however, we will also indicate the solution of the first integral as given by Nagaoka.

Put

$$\cos \theta = \alpha s + \beta \quad (13)$$

then

$$\begin{aligned} -\sin \theta d\theta &= \alpha ds \\ \sin^2 \theta &= 1 - (\alpha s + \beta)^2 = -\alpha^2 s^2 - 2\alpha\beta s + (1 - \beta^2) \\ A^2 + a^2 + z^2 - 2Aa \cos \theta &= [(a^2 + A^2 + z^2 - 2aA\beta) - 2aA\alpha s] \end{aligned}$$

and therefore,

$$\begin{aligned} \sin \theta \sqrt{A^2 + a^2 + z^2 - 2Aa \cos \theta} &= \\ \sqrt{[A^2 + a^2 + z^2 - 2aA\beta - 2aA\alpha s][-\alpha^2 s^2 - 2\alpha\beta s + (1 - \beta^2)]} \end{aligned}$$

In order that the cubic under the radical may assume the canonical Weierstrassian form

$$\sqrt{4(s - e_1)(s - e_2)(s - e_3)} = \sqrt{4s^3 - g_2s - g_3} = \sqrt{S} \quad (14)$$

the following conditions must be satisfied

$$\begin{aligned} 2aA\alpha^3 &= 4 \\ e_1 + e_2 + e_3 &= 6aA\beta\alpha^2 - \alpha^2(a^2 + A^2 + z^2) = 0 \end{aligned}$$

which give

$$\alpha = \left(\frac{2}{Aa}\right)^{\frac{1}{3}}, \quad \beta = \frac{A^2 + a^2 + z^2}{6Aa} \quad (15)$$

$$A^2 + a^2 + z^2 - 2Aa \cos \theta = \frac{4}{\alpha^3}(2\beta - \alpha s)$$

whence

$$\begin{aligned} \frac{1}{4}g_2 &= \frac{3\beta^2 + 1}{\alpha^2} & \frac{1}{4}g_3 &= \frac{2\beta(\beta^2 - 1)}{\alpha^3} \\ e_1 &= \frac{2\beta}{\alpha} & e_2 &= \frac{1 - \beta}{\alpha} & e_3 &= -\frac{(1 + \beta)}{\alpha} \end{aligned} \quad (16)$$

We have accordingly for the integrand of the first integral

$$\frac{\cos \theta \, d\theta}{(A^2 + a^2 + z^2 - 2Aa \cos \theta)^{\frac{3}{2}}} = - \frac{\alpha ds(\alpha s + \beta)}{\sqrt{S} \cdot \frac{4}{\alpha^3} (2\beta - \alpha s)}$$

and for the limits we notice that, when

$$\theta = \pi, \alpha s + \beta = -1 \text{ and therefore } s = e_3$$

$$\theta = 0, \alpha s + \beta = 1 \text{ and therefore } s = e_2$$

Introducing these conditions, the first integral becomes

$$I_1 = 4\pi A a \int_{e_2}^{e_3} \left[- \frac{\alpha^4}{4} \frac{(\alpha s + \beta)}{(2\beta - \alpha s)} \cdot \frac{ds}{\sqrt{S}} \right] \quad (18)$$

Introducing the Weierstrassian function, pu which is defined by

$$\text{the equations } u = \int_s^\infty \frac{ds}{\sqrt{S}} \text{ and } s = pu \quad (19)$$

and remembering that when $s = e_3$, $u = \omega_3$; and when $s = e_2$, $u = \omega_2$, the integral becomes

$$I_1 = -4\pi A a \int_{\omega_3}^{\omega_2} \frac{\alpha^4}{4} \frac{(\alpha pu + \beta)}{(\alpha pu - 2\beta)} du \quad (20)$$

The integrand may be written

$$1 + \frac{3\beta}{\alpha pu - 2\beta} = 1 + \frac{3}{2} \frac{e_1}{pu - e_1} \quad (21)$$

which, taking into account the relations

$$\left. \begin{aligned} \frac{1}{pu - e_1} &= - \frac{e_1}{(e_1 - e_2)(e_1 - e_3)} + \frac{p(u \pm \omega_1)}{(e_1 - e_2)(e_1 - e_3)} \\ \omega_2 - \omega_3 &= \omega_1 \end{aligned} \right\} \quad (22)$$

$$\int_{\omega_3}^{\omega_2} p(u \pm \omega_1) du = -\eta_1$$

gives finally

$$\frac{I_1}{4\pi A a} = - \frac{\alpha^4}{4} \left[\omega_1 - \frac{3}{2} \frac{e_1}{(e_1 - e_2)(e_1 - e_3)} (\eta_1 + e_1 \omega_1) \right] \quad (23)$$

From what has gone before, the second integral I_2 can be shown to be equivalent to

$$I_2 = \frac{\alpha^7}{16} \int_{\omega_3}^{\omega_2} \frac{[\alpha \mathbf{p}u + \beta]}{[\alpha \mathbf{p}u - 2\beta]^2} du \quad (24)$$

The integrand may be written

$$\frac{1}{\alpha} \frac{1}{\mathbf{p}u - e_1} + \frac{3}{2} \frac{e_1}{\alpha} \frac{1}{(\mathbf{p}u - e_1)^2}$$

and the first term can be integrated by (22). For the second term we make use of the relations

$$\left. \begin{aligned} \frac{(e_1 - e_2)^2(e_1 - e_3)^2}{(\mathbf{p}u - e_1)^2} &= \mathbf{p}^2(u \pm \omega_1) - 2e_1\mathbf{p}(u \pm \omega_1) + e_1^2 \\ \mathbf{p}^2(u + \omega_1) &= \frac{1}{6} \mathbf{p}''u + \frac{g_2}{12} \end{aligned} \right\} \quad (25)$$

which, remembering that $\mathbf{p}'\omega_3 = 0$, and $\mathbf{p}'\omega_2 = 0$ gives, finally, for the second term

$$(e_1 - e_2)^2(e_1 - e_3)^2 \int_{\omega_3}^{\omega_2} \frac{du}{(\mathbf{p}u - e_1)^2} = \frac{g_2}{12} \omega_1 + 2e_1\eta_1 + e_1^2\omega_1$$

so that the integral I_2 becomes

$$I_2 = \frac{\alpha^6}{16} \left[-\frac{(\eta_1 + e_1\omega_1)}{(e_1 - e_2)(e_1 - e_3)} + \frac{3}{2} \frac{e_1}{(e_1 - e_2)^2(e_1 - e_3)^2} \left(\frac{g_2}{12} \omega_1 + 2e_1\eta_1 + e_1^2\omega_1 \right) \right] \quad (27)$$

The integrated form of equation (12) is therefore found to be

$$\begin{aligned} \frac{1}{4\pi Aa} \frac{\partial^2 M}{\partial z^2} &= -\frac{\alpha^4}{4} \left[\omega_1 - \frac{3}{2} \frac{e_1}{(e_1 - e_2)(e_1 - e_3)} (\eta_1 + e_1\omega_1) \right] \\ &- \frac{3}{16} \alpha^6 z^2 \left[-\frac{(\eta_1 + e_1\omega_1)}{(e_1 - e_2)(e_1 - e_3)} + \frac{3}{2} \frac{e_1}{(e_1 - e_2)^2(e_1 - e_3)^2} (e_1^2\omega_1 + 2e_1\eta_1 + \frac{g_2}{12} \omega_1) \right] \end{aligned} \quad (28)$$

For numerical calculations we have next to express the formula just derived in terms of the θ functions of Jacoby, from which the transition to the q series may readily be made.

It is convenient to write the last term of (28) in the form

$$\frac{3}{2} \frac{e_1^2}{(e_1 - e_2)^2 (e_1 - e_3)^2} \left[(\eta_1 + e_1 \omega_1) + (\eta_1 + \frac{g_2}{12} \frac{\omega_1}{e_1}) \right]$$

and then, using the relations,

$$\left. \begin{aligned} \eta_1 + e_1 \omega_1 &= -\frac{1}{4\omega_1} \frac{\theta_2''(0)}{\theta_2(0)}, \quad e_1 - e_3 = \frac{\pi^2}{4\omega_1^2} \theta_3^4(0), \quad e_1 - e_2 = \frac{\pi^2}{4\omega_1^2} \theta_0^4(0) \\ e_1 &= \frac{1}{3} \frac{\pi^2}{4\omega_1^2} [\theta_3^4(0) + \theta_0^4(0)], \quad \frac{g_2}{12} = \frac{1}{18} \frac{\pi^4}{16\omega_1^4} [\theta_0^8(0) + \theta_2^8(0) + \theta_3^8(0)] \\ \eta_1 \omega_1 &= -\frac{1}{12} \frac{\theta_1'''(0)}{\theta_1'(0)} = -\frac{1}{12} \left[\frac{\theta_0''(0)}{\theta_0(0)} + \frac{\theta_2''(0)}{\theta_2(0)} + \frac{\theta_3''(0)}{\theta_3(0)} \right] \end{aligned} \right\} (29)$$

we find, finally, that the condition which must be satisfied by z_m becomes

$$\begin{aligned} \frac{3}{16} \alpha^6 z_m^2 \left[\frac{4}{\pi^4} \frac{\theta_2''}{\theta_2} \frac{1}{(\theta_3^4 + \theta_0^4)} - \frac{2}{9\pi^4} \left(\frac{1}{\theta_0^4} + \frac{1}{\theta_3^4} \right) \left(\frac{\theta_0''}{\theta_0} + 4 \frac{\theta_2''}{\theta_2} + \frac{\theta_3''}{\theta_3} \right) \right. \\ \left. - \frac{2}{9\pi^2} \left(1 - \frac{\theta_0^4}{\theta_3^4} - \frac{\theta_3^4}{\theta_0^4} \right) \right] = -\frac{\alpha^4}{\omega_1^2} \left[\frac{\theta_3^4 \theta_0^4}{(\theta_0^4 + \theta_3^4)} + \frac{1}{2\pi^2} \frac{\theta_2''}{\theta_2} \right] \end{aligned} \quad (30)$$

all the θ functions being to argument zero.

To find an expression involving q functions we make use of the well known relations

$$\begin{aligned} \theta_0 &= 1 - 2q + 2q^4 - 2q^9 + \dots + (-1)^n 2q^{n^2} + \dots \\ \theta_3 &= 1 + 2q + 2q^4 + 2q^9 + \dots + 2q^{n^2} + \dots \\ \theta_2 &= 2q^{\frac{1}{2}} [1 + q^2 + q^6 + q^{12} + \dots + q^{n(n+1)} + \dots] \end{aligned} \quad (31)$$

The substitution of these values gives rise to long and tedious reductions and only the final result will be given. This is

$$\begin{aligned} z_m^2 &= \frac{Aa}{20q} \left[1 - 36q^2 + 722q^4 - 11208q^6 + 155487q^8 \right. \\ &\quad \left. - 2065492q^{10} + 27054638q^{12} \right. \\ &\quad \left. - 353044772q^{14} + 4603353515q^{16} - \dots \right] \end{aligned} \quad (32)$$

or, if the square root be taken,

$$\begin{aligned} z_m &= \frac{\sqrt{Aa}}{2\sqrt{5q}} \left[1 - 18q^2 + 199q^4 - 2022q^6 + 21547q^8 \right. \\ &\quad \left. - 242522q^{10} + 2829828q^{12} \right. \\ &\quad \left. - 33755570q^{14} + 408424637q^{16} - \dots \right] \end{aligned} \quad (33)$$

Substituting this value of z_m in equation (5) we derive for the maximum force (putting for q the value q_0 corresponding to z_m)

$$\begin{aligned} F_m &= \frac{96}{\sqrt{5}} \pi^2 q_0^2 \left[1 + 2q_0^2 + 64q_0^4 - 252q_0^6 + 4882q_0^8 \right. \\ &\quad \left. - 50480q_0^{10} + 631392q_0^{12} \right. \\ &\quad \left. - 7604902q_0^{14} + 93488433q_0^{16} - \dots \right] \end{aligned} \quad (34)$$

In all the equations thus far given the elliptic functions have had as semiperiods the quantities ω_1 and ω_3 , respectively real and pure imaginary, and $q = e^{i\pi\tau}$, where $\tau = \frac{\omega_3}{\omega_1}$ and $i = \sqrt{-1}$. It can, however, be shown that a given expression can be made to depend also on elliptic functions having for periods ω_3 and $-\omega_1$, respectively real and imaginary, if attention be paid to the following points. The invariants g_2 and g_3 , and the quantity e_2 are unchanged by the transformation. The remaining quantities e_1 , e_3 , k , k' , τ , $q = e^{i\pi\tau}$ go into the following, respectively, e_3 , e_1 , k' , k , $\tau_1 = -\frac{1}{\tau}$, $q' = e^{i\pi\tau_1}$.

We may, consequently, write (28) in the form

$$\begin{aligned} \frac{1}{4\pi Aa} \frac{\partial^2 M}{\partial z^2} = & -\frac{\alpha^4}{4} \frac{1}{\omega_3} \left[\omega_1 \omega_3 - \frac{3e_1}{2(e_1 - e_2)(e_1 - e_3)} (\eta_1 \omega_3 + e_1 \omega_1 \omega_3) \right] \\ & - \frac{3\alpha^6}{16} z_m^2 \left[-\frac{1}{\omega_3} (\eta_1 \omega_3 + e_1 \omega_1 \omega_3) \right. \\ & \left. + \frac{3}{2} \frac{e_1^2}{(e_1 - e_2)^2 (e_1 - e_3)^2} \frac{1}{\omega_3} \left\{ 2\eta_1 \omega_3 + e_1 \omega_1 \omega_3 + \frac{g_2}{12} \frac{\omega_1 \omega_3}{e_1} \right\} \right] \end{aligned} \quad (34)$$

and for the transformation to θ functions introduce into this the relations

$$\begin{aligned} \eta_1 \omega_3 &= \omega_1 \eta_3 + i \frac{\pi}{2}, & \frac{\omega_1}{\omega_3} &= \frac{1}{\pi i} \log_e \frac{1}{q_1} \\ \eta_1 \omega_3 + e_1 \omega_1 \omega_3 &= \frac{\omega_1}{\omega_3} \left(\omega_3 \eta_3 + e_1 \omega_3^2 \right) + \frac{\pi i}{2} \\ e_1 \omega_3^2 + \omega_3 \eta_3 &= -\frac{1}{4} \frac{\theta_0'' (0/\tau_1)}{\theta_0 (0/\tau_1)} \\ e_1 - e_2 &= -\frac{\pi^2}{4\omega_3^2} \theta_2^4 (0/\tau_1), & e_1 - e_3 &= -\frac{\pi^2}{4\omega_3^2} \theta_0^4 (0/\tau_1) \\ e_1 &= -\frac{1}{3} \frac{\pi^2}{4\omega_3^2} \left[\theta_2^4 (0/\tau_1) + \theta_3^4 (0/\tau_1) \right] \\ g_2 &= \frac{2}{3} \left(\frac{\pi^2}{2\omega_3} \right)^4 \left[\theta_0^8 (0/\tau_1) + \theta_2^8 (0/\tau_1) + \theta_3^8 (0/\tau_1) \right] \end{aligned} \quad (35)$$

which give

$$\begin{aligned} & \frac{3z_m^2}{16A^2a^2} \left[-\log_e \frac{1}{q_1} \left\{ \frac{4}{\pi^4} \frac{\theta_0''}{\theta_0} \frac{1}{\theta_2^4 \theta_3^4} - \frac{2}{9} \left(\frac{1}{\theta_3^4} + \frac{1}{\theta_2^4} \right)^2 \left(\frac{\theta_2''}{\theta_2} + \frac{\theta_3''}{\theta_3} + 4 \frac{\theta_0''}{\theta_0} \right) \right. \right. \\ & \quad \left. \left. + \frac{2}{9\pi^2} \left(\frac{1}{\theta_3^4} + \frac{1}{\theta_2^4} \right) \left(1 - \frac{\theta_2^4}{\theta_3^4} - \frac{\theta_3^4}{\theta_2^4} \right) \right\} \right. \\ & \quad \left. - \frac{8}{\pi^2} \frac{1}{\theta_3^4 \theta_2^4} + \frac{8}{3\pi^2} \left(\frac{1}{\theta_3^4} + \frac{1}{\theta_2^4} \right)^2 \right] \\ & = \frac{1}{\pi^2 A a \theta_0^4} \left[\left(\frac{1}{\theta_3^4} + \frac{1}{\theta_2^4} \right) - \log_e \frac{1}{q_1} \left(1 - \frac{1}{2\pi^2} \frac{\theta_0''}{\theta_0} \left(\frac{1}{\theta_3^4} + \frac{1}{\theta_2^4} \right) \right) \right] \end{aligned} \quad (36)$$

and on substituting for these θ functions, whose argument is zero, and whose periods have the ratio τ_1 , expansions exactly like those of (31) excepting that q_1 appears in place of q , we obtain a second expression for the distance z_m corresponding to the maximum force.

$$\begin{aligned} & z_m^2 [(1 - 16q_1 + 376q_1^2 - 4672q_1^3 + 38948q_1^4 - 252192q_1^5 \\ & \quad + 1365888q_1^6 - 6463360q_1^7 + 27500946q_1^8 - \dots)] \\ & - 120q_1^2 \log_e \frac{1}{q_1} \cdot (1 - 16q_1 + 154q_1^2 - 1120q_1^3 + 6680q_1^4 \\ & \quad - 34272q_1^5 + 156268q_1^6 - \dots)] \\ & = 32aAq_1 [(1 + 24q_1 + 36q_1^2 + 384q_1^3 + 402q_1^4 + 3456q_1^5 \\ & \quad + 3064q_1^6 + 23040q_1^7 + 18351q_1^8 + \dots)] \\ & - 12q_1 \log_e \frac{1}{q_1} \cdot (1 + 2q_1 + 24q_1^2 + 28q_1^3 + 264q_1^4 \\ & \quad + 252q_1^5 + 2016q_1^6 + 1696q_1^7 + 12264q_1^8 + \dots)] \end{aligned} \quad (37)$$

It does not seem advisable to substitute this equation in (6) to obtain an expression for the maximum force F_m , but, rather, to calculate *numerical* values of z_m from (37) and to substitute this numerical value of z_m in (6).

V. USE OF THE FORMULAS

It is convenient to write the two formulas (33) and (37) just derived in the abbreviated forms

$$y_m = \frac{z_m}{A} = \sqrt{\frac{a}{A} \cdot \frac{1}{2\alpha q}} \cdot P = \sqrt{\frac{\alpha}{2\alpha q}} \cdot P \quad (33a)$$

and

$$\left(\frac{z_m}{A}\right)^2 \left[S - 12\alpha q_1^2 \log_e \frac{1}{q_1} \cdot T \right] = 32 \frac{a}{A} q_1 \left[Q - 12q_1 \log_e \frac{1}{q_1} \cdot R \right]$$

or

$$y_m^2 = \left(\frac{z_m}{A}\right)^2 = 32 \frac{a}{A} q_1 X = 32\alpha q_1 X \quad (37a)$$

respectively. Tables of values of the quantities P and X have been calculated to aid in using these formulas, and will be found in the appendix. Examples illustrating the use of these tables are given below.

In employing the formulas just derived, we are met by the difficulty that the quantity z_m depends on either q or q_1 , which, in their turn, depend on the moduli k' and k , and the latter, as (6a) shows us, involve the very quantity z_m which we wish to calculate. Since, however, the variations of q and q_1 with small changes of z_m are not important, we may solve our equations by successive approximation, and fortunately, if we know at the start, the *approximate* value of z_m , it is not a difficult matter to obtain the true value with a great deal of precision, after only a few trials. This process may be expedited by the use of suitable differential formulas, or by use of the tables of values of P and X .

1. DIFFERENTIAL FORMULAS FOR CALCULATING z_m

For small changes in z due to a given change in q , we have, in the neighborhood of ($z=z_m$), with α constant,

$$\frac{\Delta y}{y} = \frac{\Delta z}{z} = \frac{1}{2} \frac{\Delta q}{q} (1 + 72q^2 - 296q^4 + 4608q^6 - 57320q^8 + \dots) \quad (38)$$

as can be verified by differentiating (32).

Similarly, by differentiating (6a), and putting for q its approximate value $\frac{l}{2}$, we find the general expression (α constant)

$$\begin{aligned}\frac{\Delta q}{q} &= -\frac{\Delta z}{z} \cdot \frac{k^4(1+10q^4)}{4\frac{a}{A}k'^{\frac{1}{2}}(1-k')} \cdot \frac{z^2}{A^2} = -\frac{\Delta z}{z} \cdot \frac{k^2(1+k')}{4\frac{a}{A}k'^{\frac{1}{2}}} \cdot \frac{z^2}{A^2} \cdot (1+10q^4) \\ &= -\frac{\Delta y}{y} \cdot \frac{k^2(1+k')}{4\alpha k'^{\frac{1}{2}}} y^2(1+10q^4)\end{aligned}\quad (39)$$

Differentiating (37) we may also find an expression corresponding to (38). The result is not, however, simple. Writing (37) in the form $z_m^2 C_1 = 32Aaq_1 C_2$, we find (for constant α)

$$\begin{aligned}\frac{\Delta y}{y} &= \frac{\Delta z}{z} = \frac{1}{2} \frac{\Delta q_1}{q_1} \left(1 - \frac{q_1}{C_1} \frac{dC_1}{dq_1} + \frac{q_1}{C_2} \frac{dC_2}{dq_1} \right) \quad (40) \\ \frac{dC_1}{dq_1} &= -16 \left[1 - \frac{109}{2} q_1 + 996 q_1^2 - 10892 q_1^3 + 87210 q_1^4 \right. \\ &\quad \left. - 562308 q_1^5 + \dots \right] \\ &\quad - 240 q_1 \log_e \frac{1}{q_1} \left[1 - 24 q_1 + 308 q_1^2 - 2800 q_1^3 + 20040 q_1^4 \right. \\ &\quad \left. - 119952 q_1^5 + 625072 q_1^6 - \dots \right] \\ \frac{dC_2}{dq_1} &= 36 \left[1 + \frac{8}{3} q_1 + 40 q_1^2 + 54 q_1^3 + 568 q_1^4 + \frac{1784}{3} q_1^5 \right. \\ &\quad \left. + 5152 q_1^6 + \frac{13930}{3} q_1^7 + \dots \right] \\ &\quad - 12 \log_e \frac{1}{q_1} \left[1 + 4 q_1 + 72 q_1^2 + 113 q_1^3 + 1320 q_1^4 + 1512 q_1^5 \right. \\ &\quad \left. + 14112 q_1^6 + 13568 q_1^7 + 110376 q_1^8 + \dots \right] \\ C_1 &= \left[1 - 16 q_1 + 376 q_1^2 - 4672 q_1^3 + 38948 q_1^4 - 252192 q_1^5 \right. \\ &\quad \left. + 1365888 q_1^6 - \dots \right] \quad (41) \\ &\quad - 120 q_1^2 \log_e \frac{1}{q_1} \left[1 - 16 q_1 + 154 q_1^2 - 1120 q_1^3 + 6680 q_1^4 \right. \\ &\quad \left. - 34272 q_1^5 + 156268 q_1^6 - \dots \right] \\ C_2 &= \left[1 + 24 q_1 + 36 q_1^2 + 384 q_1^3 + 402 q_1^4 + 3456 q_1^5 + 3064 q_1^6 \right. \\ &\quad \left. + 23040 q_1^7 + 18351 q_1^8 + \dots \right] \\ &\quad - 12 q_1 \log_e \frac{1}{q_1} \left[1 + 2 q_1 + 24 q_1^2 + 28 q_1^3 + 264 q_1^4 + 252 q_1^5 \right. \\ &\quad \left. + 2016 q_1^6 + 1696 q_1^7 + 12264 q_1^8 + \dots \right]\end{aligned}$$

and the equation corresponding to (39) is (α constant)

$$\frac{\Delta q_1}{q_1} = \frac{\Delta z}{z} \cdot \frac{k^{\frac{3}{2}}}{4 \frac{a}{A}(1-k)} \cdot \frac{z^2}{A^2} = \frac{\Delta z}{z} \frac{k^{\frac{3}{2}}(1+k)}{4 \frac{a}{A} k'^2} \cdot \frac{z^2}{A^2} = \frac{\Delta y}{y} \frac{k^{\frac{3}{2}}(1+k)y^2}{4\alpha k'^2} \quad (42)$$

Table 3 (appendix) gives the values of $\frac{\Delta z}{z}$ and $\frac{\Delta q}{q}$ or $\frac{\Delta q_1}{q_1}$ for a number of values of α , and will be found useful in making calculations by this method.

In making a calculation of the value of z_m , using these differential formulas, the procedure is as follows: Having obtained an approximate value of z_m by methods to be described below, the calculation of q or q_1 is made (see Example 1), and the values thus found used in (33) or (37). In general, a somewhat more accurate value of z_m will thus be obtained, which will differ from the original by a certain fractional part of the whole, and this difference is then to be taken as $\frac{\Delta z}{z}$ in formula (39) or (42), according as (33) or (37) is being used. We obtain thus the fractional change in q or q_1 corresponding to the second approximation of z_m . With this value of $\frac{\Delta q}{q}$ or $\frac{\Delta q_1}{q_1}$ we can at once compute from (38) or (40) a third approximation of z_m , and by repeating this process we rapidly approximate the true value of z_m . For this process we need calculate the coefficients in the differential formulas *once only*, since they do not need to be known with any great accuracy, and in general the values interpolated from Table 3 will be sufficient.

From an examination of this process it will be evident that each successive approximation of z_m differs from the preceding by a fractional amount which is related to the corresponding fractional difference immediately preceding by a constant factor f , which is less than unity. Or, otherwise expressed, if $z_1, z_2, \dots, z_m$ are the successive approximations and

$$\frac{\Delta z}{z} = \frac{z_2 - z_1}{z_1} \text{ then } \frac{z_m - z_1}{z_1} = \frac{\Delta z}{z} [1 + f + f^2 + \dots + f_m] = \frac{\Delta z}{z} \frac{1}{1 - f} \text{ in its}$$

limit. The factor f may be taken from Table 3.

Having obtained the final value of z_m , a check calculation should be made to see whether this value satisfies the main formula (33) or (37), which by the procedure just outlined does not therefore need to be used but twice, including the check calculation.

2. APPROXIMATE FORMULAS

Obviously the amount of time and labor necessary to obtain a precise value of z_m by the preceding formulas depends, in large measure, on the closeness with which the first approximation, which is made the basis of the calculation, approaches the true value. It is, therefore, very desirable that, before using formulas (33) or (37), a value of z_m may be found which is within, say, one part in a thousand of the truth.

The Rayleigh formula (8) or formula (9) is very satisfactory for this purpose for small values of $\frac{a}{A}$; for the important case of $\frac{a}{A} = \frac{1}{2}$, it is, however, one per cent in error, and for still larger values of $\frac{a}{A}$ its insufficiency becomes such as to render its use entirely unsatisfactory.

For values of $\frac{a}{A}$ for which a is nearly equal to A , the following development of (37) is of assistance.

Putting $\frac{a}{A} = 1 - \delta$, $\frac{z}{A} = \gamma$, we find from (6a) the approximation (holding for small values of δ)

$$q_1 = \frac{(\delta^2 + \gamma^2)}{64} \left[1 + \delta + \frac{1}{8} (\delta^2 + \gamma^2) \right] \quad (43)$$

and (37) may be written with sufficient accuracy

$$\gamma_m^2 = \left(\frac{z_m}{A} \right)^2 = 32 \frac{a}{A} q_1 \left[\frac{1 + 24q_1 - 12q_1 \log_e \frac{1}{q_1}}{1 - 16q_1} \right] \quad (44)$$

To obtain a starting point for the use of (43) and (44), we neglect for the moment the last term in (43). The main term in (44) then becomes $\gamma_m^2 = 32 (1 - \delta) \frac{(\delta^2 + \gamma_m^2)}{64} (1 + \delta)$, whence

$$\gamma_m = \delta \sqrt{\frac{1 - \delta^2}{1 + \delta^2}} \quad (45)$$

Starting with this value in (43), we calculate a value of q_1 to be used in (44), and thus a second approximation of y_m , and so on. Very little work is necessary to obtain a very satisfactory approximation to y_m , since the correction term in (44) need be calculated but once, and a very moderate accuracy in the value of $\log_e \frac{1}{q_1}$ will suffice.

The range of usefulness of (43) and (44) is not, however, great, and does not include those cases which are most important in practice. For the especially interesting case, where $\frac{a}{A}$ is neither

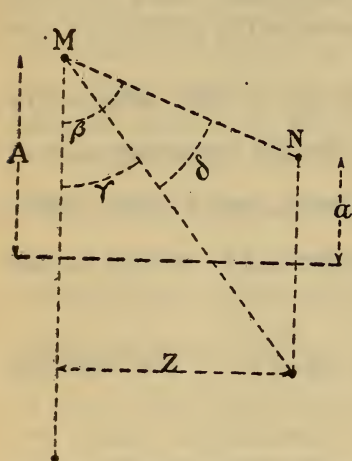


FIG. 1.—Definition of the angle δ

small nor near to unity, the following empirical method has proved very useful. Having, so far as the author has been able to determine, no simple physical significance, it is to be regarded as merely a useful coincidence.

If in Fig. 1 we pass a plane through the common axis of the two circles, and calculate the angle δ , subtended by that diameter NP of the small circle which lies in this plane (as viewed from the point M of the larger circle, lying in the same plane), it will be found that the critical distance z_m has such a value as to make

the angle $\psi = \frac{A}{a} \delta$ roughly the same, whatever the value of $\frac{a}{A}$.

This angle ψ has a value of 45° for $\alpha = \frac{a}{A} = 1$, increases to a value of about $46^\circ 30'$ for $\alpha = 0.7$, and falls with decreasing values of α to about $45^\circ 50'$ for $\alpha = 0$.

To aid in finding this angle for any given value of $\frac{a}{A}$, Table 4 will be useful. In this the angle ψ has been calculated for a number of values of $\frac{a}{A} = \alpha$, for which the distance z_m was calculated to within a part in a million by the methods already

described. To aid in interpolation the curve, Fig. 2, connecting the values in this table, has been plotted.

Having taken from the curve, or by interpolation from the table the value of the angle ψ for any other value of α not included in the table, we derive the approximate value of $y_m = \frac{z_m}{A}$ desired, by means of the following equations.

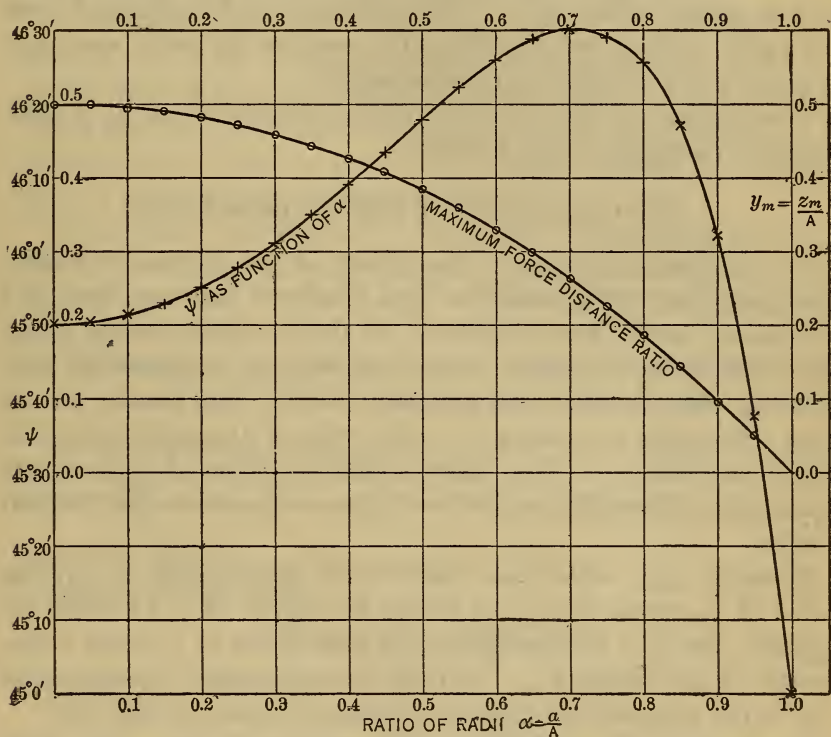


FIG. 2.—Giving the angle ψ as a function of the ratio of the radii. Useful in obtaining a first approximation to y_m

Putting

$$\alpha = \frac{a}{A}, \psi = \frac{\delta}{\alpha}, M = \tan \delta, \frac{z}{A} = y$$

$$M = \tan (\beta - \gamma) = \frac{2 \alpha y}{(1 - \alpha^2) + y^2} \quad (46)$$

$$\therefore y_m = \frac{\alpha}{M} - \sqrt{\frac{\alpha^2}{M^2} - (1 - \alpha^2)} = \frac{\alpha}{M} \left[1 - \sqrt{1 - \frac{M^2}{\alpha^2} (1 - \alpha^2)} \right]$$

the angles β and γ being those shown in Fig. 1.

A study of the equation

$$\frac{\Delta y}{y} = \frac{\Delta M}{M} \left[\frac{1}{\sqrt{1 - \frac{M^2}{\alpha^2}(1 - \alpha^2)}} \right] \quad (y = y_m)$$

shows, that in the most unfavorable case, to obtain y_m correct to one part in 1,000, the angle ψ taken from the curve must be correct to within about 30". This accuracy can be obtained from the curve without serious difficulty, even in the worst case, and in general this accuracy can be exceeded.

This method of obtaining a first approximation to y_m is illustrated in examples 4 and 5 below.

3. THE CALCULATION OF THE MAXIMUM FORCE

The distance z_m between the planes of the circular currents, corresponding to the condition that the force between them is a maximum, having been calculated by the formulas already given, the value of the maximum force itself may be computed by substituting the value of z_m in formula 5 or 6. The former will be used principally for values of α (the ratio of the radii) lying between zero and 0.5. The latter holds for circles of more nearly equal radii, although the two formulas overlap somewhat in their ranges.

Formula (34), which was derived by substituting in (5) the value of y_m given by (33), is useful for values of α to which (5) applies. In (34) the quantity q_0 is that value of q which corresponds to the distance z_m . It must be emphasized, however, that (34) is not a *general* formula for the force, as are (5) and (6).

The method of calculation of the maximum force from these three formulas is illustrated in examples 10 and 11, such a value of α having been chosen that all three formulas are applicable to the same case, thus giving a check on the accuracy of the three formulas.

Table 4 gives the values of y_m and the maximum force F_m for values of α from 0.05 to 0.95 in steps of 0.05. In computing these values, seven place logarithms were used so that the values given should not be in error by more than a part in a million.

Since, in a practical case, the coils would be designed so that their radii should approximate some round value, the data in Table 4 may be regarded as fundamental, and the usefulness of this table is further enhanced by the fact that the values of y_m and F_m for values of α differing slightly from these fundamental values, may be obtained very simply from the data in the table without recourse to the main formulas (33) and (37).

For example, if a pair of coils, constructed to have radii in the ratio of 1 to 2, are found, as a result of measurement, to have an actual ratio of the radii greater than one-half by 0.005, the maximum force for this pair of coils can be found by applying to the value given in Table 4 for $\alpha=0.5$, a factor, which is a function both of the difference 0.005 and of α .

To calculate this factor it is necessary to determine what relation exists between a small change in α and the changes in y_m and F_m corresponding. This matter is taken up in the next section.

VI. VARIATION FORMULAS FOR THE MAXIMUM FORCE F_m AND THE MAXIMUM DISTANCE RATIO y_m

To determine the change in y_m or F_m for a small fractional change $\frac{\Delta\alpha}{\alpha}$ in the ratio of the radii, we differentiate equations (33a) and (37a).

Equation (33a) gives

$$\frac{\Delta y_m}{y_m} = \frac{1}{2} \frac{\Delta\alpha}{\alpha} - \frac{1}{2} \frac{\Delta q}{q} + \frac{\Delta P}{P} \quad (46)$$

Writing

$$\Delta q = \frac{\partial q}{\partial \alpha} \Delta\alpha + \frac{\partial q}{\partial y_m} \Delta y_m$$

$$\frac{\Delta P}{P} = \frac{1}{P} \frac{\partial P}{\partial q} \Delta q, \quad \frac{\partial q}{\partial \alpha} = \frac{\partial q}{\partial k'} \cdot \frac{\partial k'}{\partial \alpha}, \quad \frac{\partial q}{\partial y_m} = \frac{\partial q}{\partial k'} \cdot \frac{\partial k'}{\partial y_m}$$

and using (6a) to evaluate the differential coefficients, we have

$$\Delta q = \frac{4K'\alpha}{Z} \left[y_m^2 \cdot \frac{\Delta y_m}{y_m} - \frac{\lambda}{2} \cdot \frac{\Delta\alpha}{\alpha} \right]$$

$$\frac{1}{P} \frac{\partial P}{\partial q} = -36q \left[1 - \frac{37}{9} q^2 + 64q^4 - \frac{7165}{9} q^6 + \frac{93974}{9} q^8 - \dots \right] \quad (47)$$

which, substituted in (46) give the relation

$$\frac{\Delta y_m}{y_m} \left[1 + \frac{4K'\alpha}{Z} y_m^2 \left(\frac{1}{2q} - \frac{1}{P} \frac{dP}{dq} \right) \right] = \frac{1}{2} \frac{\Delta \alpha}{\alpha} \left[1 + \frac{4K'\alpha}{Z} \lambda \left(\frac{1}{2q} - \frac{1}{P} \frac{dP}{dq} \right) \right] \quad (48)$$

where

$$K' = -\frac{(1 + 10q^4)}{2\sqrt{k'}(1 + \sqrt{k'})^2}, \quad \lambda = 1 - \alpha^2 + y_m^2 \quad (49)$$

$$Z = [(1 + \alpha)^2 + y_m^2]^{\frac{3}{2}} \sqrt{(1 - \alpha)^2 + y_m^2}$$

For the change in the maximum force

$$\frac{\Delta F_m}{F_m} = \frac{1}{F_m} \cdot \frac{\partial F_m}{\partial q} \cdot \Delta q \quad (50)$$

Writing $\frac{\Delta y_m}{y_m} = \frac{H}{2} \frac{\Delta \alpha}{\alpha}$ in the equation for Δq previously derived, and differentiating (34) to obtain $\frac{1}{F_m} \frac{\partial F_m}{\partial q}$, the final expression becomes

$$\frac{\Delta F_m}{F_m} = \frac{4K'\alpha}{Zq} (y_m^2 H - \lambda) (1 + 2q^2 + 124q^4 - 1132q^6 + 14360q^8 - \dots) \cdot \frac{\Delta \alpha}{\alpha} \quad (51)$$

in which the factor H is to be calculated by formula (48).

The expressions (48) and (51) are to be used for values of α up to about 0.5. To cover the range of larger values, we start from (37a), which gives on differentiation

$$\frac{\Delta y_m}{y_m} = \frac{1}{2} \left[\frac{\Delta C_2}{C_2} - \frac{\Delta C_1}{C_1} + \frac{\Delta \alpha}{\alpha} + \frac{\Delta q_1}{q_1} \right] \quad (52)$$

Writing

$$\Delta q_1 = \frac{\partial q_1}{\partial k} \cdot \frac{\partial k}{\partial \alpha} \cdot \Delta \alpha + \frac{\partial q_1}{\partial k} \cdot \frac{\partial k}{\partial y_m} \cdot \Delta y_m$$

$$K = -\frac{1}{2\sqrt{k}(1 + \sqrt{k})^2}, \quad N = \frac{\Delta C_1}{C_1} = \frac{1}{C_1} \cdot \frac{\partial C_1}{\partial q_1} \cdot \Delta q_1 \quad (53)$$

$$Y = [(1 + \alpha)^2 + y_m^2]^{\frac{3}{2}}, \quad M = \frac{\Delta C_2}{C_2} = \frac{1}{C_2} \cdot \frac{\partial C_2}{\partial q_1} \cdot \Delta q_1$$

and using (6a) to carry out the differentiations indicated in (53),

$$\Delta q_1 = \frac{K\sqrt{\alpha}}{Y} \left[\lambda \frac{\Delta \alpha}{\alpha} - 2y_m^2 \cdot \frac{\Delta y_m}{y_m} \right] \quad (54)$$

from whence we find immediately

$$\frac{\Delta y_m}{y_m} \left[1 + \frac{K\sqrt{\alpha}}{Y} y_m^2 \left(M - N + \frac{1}{q_1} \right) \right] = \frac{1}{2} \frac{\Delta \alpha}{\alpha} \left[1 + \frac{K\sqrt{\alpha}}{Y} \lambda \left(M - N + \frac{1}{q_1} \right) \right] \quad (55)$$

in which the quantities M and N are to be calculated from (53) and (41).

Equation (6) for the maximum force may be written $F_m = \frac{\pi y_m \Phi}{16 q_1 \sqrt{\alpha}}$ which gives on differentiation

$$\frac{\Delta F_m}{F_m} = \frac{\Delta y_m}{y_m} - \frac{1}{2} \frac{\Delta \alpha}{\alpha} - \frac{\Delta q_1}{q_1} + \frac{\Delta \Phi}{\Phi} \quad (56)$$

Putting

$$\frac{\Delta \Phi}{\Phi} = \frac{1}{\Phi} \frac{d\Phi}{dq_1} \cdot \Delta q_1 = \Pi \Delta q_1 \quad (57)$$

and writing as before, $\frac{\Delta y_m}{y_m} = \frac{H}{2} \cdot \frac{\Delta \alpha}{\alpha}$, equations (56) and (57) become

$$\begin{aligned} \frac{\Delta F_m}{F_m} &= \frac{1}{2} (H - 1) \frac{\Delta \alpha}{\alpha} + \left(\Pi - \frac{1}{q_1} \right) \Delta q_1 \\ &= \frac{\Delta \alpha}{\alpha} \left[\frac{1}{2} (H - 1) + \left(\Pi - \frac{1}{q_1} \right) \frac{K\sqrt{\alpha}}{Y} (\lambda - Hy_m^2) \right] \end{aligned} \quad (58)$$

To calculate Π we have

$$\begin{aligned} \frac{d\Phi}{dq_1} &= 24(1 - 21q_1 + 184q_1^2 - 1089q_1^3 + 5160q_1^4 - \dots) \\ &\quad - 12 \log_e \frac{1}{q_1} \cdot (1 - 20q_1 + 180q_1^2 - 1200q_1^3 + 6500q_1^4 - \dots) \\ \Phi &= (1 + 12q_1 - 192q_1^2 + 1232q_1^3 - 5634q_1^4 + \dots) \\ &\quad - 12q_1 \log_e \frac{1}{q_1} \cdot (1 - 10q_1 + 60q_1^2 - 300q_1^3 + 1300q_1^4 - \dots) \end{aligned} \quad (59)$$

These formulas, although somewhat complicated in appearance, are not difficult to use, especially since most of the quantities involved enter also in the main formulas.

The factors η and ϵ , defined by the equations

$$\begin{aligned}\frac{\Delta y_m}{y_m} &= \eta \frac{\Delta \alpha}{\alpha} \\ \frac{\Delta F_m}{F_m} &= \epsilon \frac{\Delta \alpha}{\alpha}\end{aligned}\tag{60}$$

have been calculated by means of the equations just developed, for the same values of α as in Table 4, and are given in Table 5. Values of η and ϵ for other values of α may be derived by interpolation. From the method of derivation of the formulas for η and ϵ , it is evident that these values hold only for very small values of $\frac{\Delta \alpha}{\alpha}$. For differences, $\frac{\Delta \alpha}{\alpha}$, greater than about 0.001, the values taken from the table must be corrected by a factor whose value is determined from the following considerations.

Table 4 gives the value F_0 of the maximum force for a certain value α_0 of the ratio of the radii. It is desired to find the value of the maximum force $F_1 = F_0 \left(1 + \frac{\Delta F}{F_0} \right)$ corresponding to a slightly different ratio of the radii α_1 , connected with the former ratio by the relation $\alpha_1 = \alpha_0 \left(1 + \frac{\Delta \alpha}{\alpha_0} \right)$.

For an infinitely small increment of α_0 , $d\alpha$, the fractional change in the value of the maximum force is $\frac{dF}{F_0} = \epsilon_0 \frac{d\alpha}{\alpha_0}$, the value of ϵ_0 being that taken from Table 5 for the argument α_0 . Similarly, in the neighborhood of α_1 , $\frac{dF}{F_1} = \epsilon_1 \frac{d\alpha}{\alpha_1}$, ϵ_1 being obtained from Table 5 by direct interpolation, using the argument α_1 . For the *finite* difference $\Delta \alpha = \alpha_1 - \alpha_0$, the corresponding difference ΔF is

$$\frac{\Delta F}{F_0} = \frac{1}{F_0} \int_{\alpha=\alpha_0}^{\alpha=\alpha_1} dF = \frac{1}{\alpha_0} \int_{\alpha_0}^{\alpha_1} \epsilon d\alpha$$

where the coefficient ϵ is a function of α . We may approximate

the value of this integral closely, if we use the average value of ϵ over the interval $\Delta\alpha$. This average value is, however, not the simple mean of ϵ_0 and ϵ_1 , since $\epsilon_1 = \frac{\alpha_1}{F_1} \frac{dF}{d\alpha}$ and that portion of the integral contributed by the element $d\alpha$ in the neighborhood of α_1 is not $\epsilon_1 \frac{d\alpha}{\alpha_1}$, but $\epsilon_1 \frac{\alpha_0}{\alpha_1} \frac{F_1}{F_0}$ because in the original statement of the problem, fractional increments of F_m and α are to be referred to F_0 and α_0 , the fundamental values given in Table 4.

Taking these facts into account in finding the average value of ϵ , we find as an approximation to the integral the expression

$$\frac{\Delta F}{F_0} = \frac{1}{2} \frac{\Delta\alpha}{\alpha_0} \left[\epsilon_0 + \frac{F_1}{F_0} \cdot \frac{\alpha_0}{\alpha_1} \cdot \epsilon_1 \right] \quad (61)$$

which is a very close approximation to the truth for values of $\frac{\Delta\alpha}{\alpha_0}$ as great as 0.01, over the range of values of α for which interpolation in Table 5 may be accurately made.

Entirely similar considerations enable to be found for the change in γ_m corresponding to a small change in α , the equation

$$\frac{\Delta\gamma}{\gamma_0} = \frac{1}{2} \frac{\Delta\alpha}{\alpha_0} \left[\eta_0 + \frac{\gamma_1}{\gamma_0} \cdot \frac{\alpha_0}{\alpha_1} \cdot \eta_1 \right] \quad (62)$$

These formulas are illustrated in examples 13 and 14, where the constants are calculated for the coils of the Bureau of Standards current balance, and the results compared with the values calculated by the methods described in the original article.

VII. VARIATION OF THE FORCE FOR SMALL DISPLACEMENTS OF THE COILS FROM THE POSITION OF MAXIMUM FORCE

From Maxwell's formula (3) it is evident that the force corresponding to positions of the coils in the neighborhood of the critical distance bears to the maximum value of the force a relation which may be expressed by the power series

$$F = F_m [1 - \gamma (z_m - z)^2 + \delta (z_m - z)^3 - \dots] \quad (63)$$

in which z is the actual distance between the planes of the coils, z_m the distance for which the force has its maximum value F_m ,

and γ and δ are constants. The term in the first power of the displacement ($z_m - z$) is lacking.

Remembering that the force is a function of the ratio of the radii alone, we may write

$$\frac{\Delta F}{F_m} = -c \left(\frac{\Delta y}{y_m} \right)^2 + d \left(\frac{\Delta y}{y_m} \right)^3 - \dots \quad (64)$$

where c and d are positive constants depending on α alone. This formula holds with a good deal of accuracy for values of $\frac{\Delta y}{y_m}$ as great as 0.04.

The following table will allow of the interpolation of the value of the constant c with an accuracy sufficient in practically all cases:

Values of the Constant c as a Function of α

α	c
0.1	0.805
0.3	0.755
0.5	0.690
0.7	0.610
0.9	0.510

The constant d is sensibly equal to 0.5 for values of α ranging from 0.1 to 0.9. These values of c and d were found by actually calculating the force, by the methods already developed, for various values of α , and for differences $\frac{\Delta y}{y_m}$ of ± 0.01 and ± 0.04 .

The value of $\frac{\Delta F}{F_m}$ corresponding to a negative displacement $\frac{-\Delta y}{y_m}$ differs from that produced by an equal positive displacement $\frac{\Delta y}{y_m}$ by an amount which depends on d alone. The average value of $\frac{\Delta F}{F_m}$ corresponding to equal positive and negative displacements $\frac{\Delta y}{y_m}$ and $\frac{-\Delta y}{y_m}$ is a measure of the value of c alone. The values of c and d here given show how slowly the force varies in the neighborhood of the position of maximum force.

VIII. BEST VALUES OF RADII OF THE COILS

The formulas and methods above developed furnish a solution of the general problem of finding the maximum force between two circular currents with any given ratio of their radii. The practically important question, What ratio of the radii is best in designing a Rayleigh current balance? may also be answered in a general way from a consideration of the numerical values in the tables here presented.

The conditions which should be fulfilled by a current balance are broadly as follows:

1. The force should be large enough to be accurately measurable.
2. The number of turns on the coils should be kept as small as possible in order to provide that the resistance of the coils may be small. Otherwise the heating of the coils may cause considerable changes in the dimensions of the coils, as well as air currents which will affect the accuracy of the weighing.
3. The cross section of the coils should be made small relatively to the distance between the windings of the fixed and movable coils.

Naturally the best design will be one of compromise, since these conditions are to some extent incompatible; and, as is usual, quite a latitude in the design is allowable without materially detracting from the performance of the balance.

Large values of α (radius of the moving coil nearly equal to that of the fixed coil) give large values of the force per turn per unit current. This advantage, however, is counterbalanced by the necessity of keeping the cross section of the coils small; for, as may be seen from Table 4, the distance z_m is, in the cases supposed, so small that very moderate values of the area of cross section will give rise to large corrections for the latter, with consequent difficulty in determining its value. Further, the change in the force for small changes in the radii and for small displacements of the coils from the position of maximum force are much larger with coils of nearly equal radii (α large) than for coils of quite different radii (α small).

These disadvantages, combined with the undesirability of using such large moving coils as postulate a long arm balance, render impracticable the employment of moving coils with radii nearly equal to the radius of the fixed coil.

The opposite case of a balance whose moving coil is small relatively to the fixed coil, although free from the difficulties of the previous case, suffers under the disadvantage of a small value of the force per unit current per turn, so that either the number of turns on the coils or else the value of the currents must be increased to compensate for this difficulty. This renders probable in the former case a troublesome correction for the area of cross section of the coils, and in the latter difficulties in the manipulation of the balance, as a result of air currents or, at least, considerable changes in the dimensions of the coils as a result of undue heating of the coils.

From these considerations one is lead to expect that a suitable design will avoid either extreme, and this is confirmed by the numerical results given in Table 4. Starting with very small values of the maximum force (per unit current per turn in each coil) for values of α about 0.1, the value of F_m increases rapidly with increase of α , so that it is not difficult to obtain a sufficiently large value of the force with a moving coil having a radius about one-half as large as that of the fixed coils, and without encountering serious difficulty from the disadvantages which beset the use of relatively small moving coils on the one hand and relatively large moving coils on the other.

These points are illustrated in the following example, in which are tabulated the principal constants of five current balances, each of which is supposed to have a fixed coil of 25 cm radius, the radii of the moving coils in the different cases being taken of various values. The nomenclature is the same as in the preceding discussion, except that there are introduced the additional quantities d_m , the distance between the windings of the coils in a radial plane and Δ the fractional change in the value of the force for a displacement of 1 mm from the position of maximum force.

$\alpha =$	0.1	0.3	0.5	0.7	0.9
a	2.5	7.5	12.5	17.5	22.5
z_m	12.39	11.47	9.59	6.61	2.446
d_m	25.68	20.92	15.75	10.00	3.50
F_m	0.1709	1.646	5.352	14.507	58.06
ϵ	2.016	2.160	2.549	3.649	9.934
δ	53×10^{-6}	60×10^{-6}	75×10^{-6}	137×10^{-6}	868×10^{-6}

An examination of these values shows that it will not be advisable to use a value of α greater than about 0.7 nor less than about 0.3. The coils of the Bureau of Standards coils were designed to have ratios of 0.4 and 0.5.

IX. EXAMPLES

EXAMPLE 1. CALCULATION OF q AND q_1 (FORMULAE 6a)

One or both of these quantities has to be calculated when the majority of the formulas of this paper are to be used. The following arrangement of the calculations has been found convenient:

Let us obtain the values of q and q_1 corresponding to the case $\alpha = \frac{a}{A} = 0.5$, $\gamma = \frac{z}{A} = 0.3875$. The formulas (6a) for k and k' may be conveniently written in the form

$$k^2 = \frac{4\frac{a}{A}}{\left(1 + \frac{a}{A}\right)^2 + \left(\frac{z}{A}\right)^2} = \frac{4\frac{a}{A}}{4\frac{a}{A} + \left[\left(1 - \frac{a}{A}\right)^2 + \frac{z^2}{A^2}\right]} = \frac{4\alpha}{4\alpha + [(1-\alpha)^2 + \gamma^2]}$$

$$k'^2 = \frac{\left(1 - \frac{a}{A}\right)^2 + \left(\frac{z}{A}\right)^2}{\left(1 + \frac{a}{A}\right)^2 + \left(\frac{z}{A}\right)^2} = \frac{\left(1 - \frac{a}{A}\right)^2 + \left(\frac{z}{A}\right)^2}{4\frac{a}{A} + \left[\left(1 - \frac{a}{A}\right)^2 + \frac{z^2}{A^2}\right]} = \frac{[(1-\alpha)^2 + \gamma^2]}{4\alpha + [(1-\alpha)^2 + \gamma^2]}$$

Here $4\alpha = 2$,

$$(1-\alpha)^2 = 0.25 \quad \log \gamma = \bar{1}.5882717$$

$$2 \log \gamma = \bar{1}.1765434$$

$$\gamma^2 = 0.15015624$$

$$(1-\alpha)^2 = 0.25$$

$$\text{Sum} = 0.40015624$$

$$k^2 = \frac{2}{2.40015624}$$

$$k'^2 = \frac{0.40015624}{2.40015624}$$

log numerator = 0.3010300	log numerator = <u>1.6022296</u>
log denominator = <u>0.3802395</u>	log denominator = <u>0.3802395</u>
log k^2 = <u>1.9207905</u>	log k'^2 = <u>1.2219901</u>
log M = <u>0.8788851</u>	log k' = <u>1.6109950</u>
Diff. = $\log \frac{l}{2} = 1.0419054$	$\log \sqrt{k'} = 1.8054975$
$5 \log \frac{l}{2} = 5.20952$	$(1 + \sqrt{k'}) = 1.6389950$
log 2 = <u>0.30103</u>	$(1 + k') = 1.4083147$
Sum = <u>5.51055</u>	$\log (1 + \sqrt{k'}) = 0.2145777$
$2\left(\frac{l}{2}\right)^5 = 0.00003240$	$\log (1 + k') = 0.1486997$
$\frac{l}{2} = 0.11012995$	log 2 = <u>0.3010300</u>
$q = 0.11016235$	$\log 2(1 + k') = 0.4497297$
$\log q = 1.0420332$	$2 \log (1 + \sqrt{k'}) = 0.4291554$
	log M = Sum = <u>0.8788851</u>

Check.

$$\begin{aligned}
 \sqrt{k'} &= 0.6389950 \\
 1 - \sqrt{k'} &= 0.3610050 & \log (1 - \sqrt{k'}) &= 1.5575132 \\
 2(1 + \sqrt{k'}) &= 3.2779900 & \log 2(1 + \sqrt{k'}) &= 0.5156076 \\
 \text{Diff.} &= \log \frac{l}{2} = 1.0419056
 \end{aligned}$$

In the above calculation q has been obtained by the use of the second of the formulas under (6a). The first relation, that involving $1 - \sqrt{k'}$, may, when $\sqrt{k'}$ is nearly equal to unity, give an accuracy much inferior to that obtained by the other formula. It is, however, useful as a check, and is so used in the calculation just given.

The procedure to be followed in calculating q_1 is, in every respect, similar to that for q , except that k takes the place of k' throughout, and that the difference between q_1 and $\frac{l_1}{2}$ is, on account of the smallness of q_1 , generally negligible.

$\log k^2 = \bar{1}.9207905$	$\log k'^2 = \bar{1}.2219901$
$\log k = \bar{1}.9603952$	$\log M = \underline{1.1651921}$
$\log \sqrt{k} = \bar{1}.9801976$	$\log \frac{l_1}{2} = \log q = \bar{2}.0567980$
$1 + \sqrt{k} = 1.9554272$	
$1 + k = 1.9128411$	since $5 \log \frac{l_1}{2} = \bar{1}\bar{0}.28399$
$\log (1 + \sqrt{k}) = 0.2912416$	$q_1 = 0.011397195$
$\log (1 + k) = 0.2816789$	
$\log 2 = \underline{0.3010300}$	
$\log 2 (1 + k) = 0.5827089$	<i>Check.</i>
$2 \log (1 + \sqrt{k}) = \underline{0.5824832}$	$\sqrt{k} = 0.9554272$
$\log M = \text{Sum} = 1.1651921$	$1 - \sqrt{k} = 0.0445728$
	$2(1 + \sqrt{k}) = \underline{3.9108544}$
	$\log (1 - \sqrt{k}) = \bar{2}.6490699$
	$\log 2(1 + \sqrt{k}) = \underline{0.5922716}$
	$\text{Sum} = \log \frac{l_1}{2} = \bar{2}.0567983$

EXAMPLE 2.—ILLUSTRATING THE CALCULATION OF $y_m = \frac{z_m}{A}$ BY (33), STARTING FROM THE APPROXIMATION BY RAYLEIGH'S FORMULA (8)

For the case $\alpha = \frac{a}{A} = 0.5$ we obtain as a first approximation $y = 0.3875$. The value of q corresponding to this value of y has been calculated in the previous example.

From Table 1 we find by interpolation that the value of P corresponding to $q = 0.11016235$ is $P = 0.807662$. The calculation of a second approximation to y by equation (33) is as follows:

$\log q = \bar{1}.0420332$	$\frac{1}{2} \log \frac{\alpha}{20q} = \bar{1}.6779534$
$\log 20 = \underline{1.3010300}$	$\log P = \bar{1}.9072297$
$\text{Sum} = 0.3430632$	$\text{Sum} = \log y = \bar{1}.5851831$
$\log \alpha = \underline{1.6989700}$	$y = 0.384754 =$
$\log \frac{\alpha}{20q} = \bar{1}.3559068$	second approximation

The calculation may now be repeated, using this second approximation for y_m to find more accurate values of k , k' , and q , and thence a third approximation to y_m , and so on till successive

approximations do not differ sensibly from the preceding. The successive approximations in the present case are, to six places of logarithms,

Approximation	$\log y$
1	$\bar{1}.588272$
2	$\bar{1}.585183$
3	$\bar{1}.584228$
4	$\bar{1}.583933$
5	$\bar{1}.583843$
6	$\bar{1}.583816$
7	$\bar{1}.583807$
8	$\bar{1}.583804$
9	$\bar{1}.583803$

The necessary labor may, however, be very much reduced by the use of the differential formulas (38) and (39). Using in (39) the second approximation of y_m already found, and the values of k , k' , and q already calculated above, there results

$$\frac{\Delta q}{q} = -0.3330 \frac{\Delta z}{z}, \quad \frac{\Delta z}{z} = -0.9215 \frac{\Delta q}{q}$$

Since we are concerned with fairly small differences, we may, without serious error, use in place of $\frac{\Delta q}{q}$ and $\frac{\Delta z}{z}$ the differences in the logarithms of the different values of q and z or y . The difference of the logarithms of the first and second approximation to y_m is 0.0030866. The formulas (38) and (39) show that the logarithm of the third approximation to y_m will differ from that of the second approximation by 0.3330 of 0.9215 of 0.0030886, and the difference between the logarithms of the fourth and third approximations will be less than this value in the ratio of the same factor, and so on for each successive approximation. Denoting this factor by f , we may find at once the factor F by which we must multiply the difference between the first and second approximations, in order to obtain the amount by which the final value differs from the first approximation. It is $F = 1 + f + f^2 + f^3 + \dots + f^n$, which gives, when n is taken indefinitely great, the value $F = \frac{1}{1-f}$. Using here the value $f = 0.30685$,

and the difference 0.0030866, the final difference 0.0044559 is found, which subtracted from the logarithm of the first approximation gives $\log y_m = \bar{1}.5838158$. This value differs slightly from the true value, for the reason that the data used in calculating the factor f does not quite apply to such a large interval as that here in question. We may, however, use this value as the starting point in obtaining the true value. Using the value $\bar{1}.5838158$ as a first approximation, formula (33) gives as a second approximation $\bar{1}.5838076$ (a difference in the logarithm of -82 in the seventh place), which, using the same value of the factor f as before, indicates that the correction to be applied is -118 in the seventh place, and that the final value is $\bar{1}.5838040$. Substituting this value in formula (33) the result obtained does not differ from this by as much as a unit in the seventh place, showing that equation (33) is satisfied and that the correct value has been found.

This method was used in obtaining the data in Table 4. In most cases, however, the first approximation used did not differ by more than one or two parts in a thousand from the true value, and for such cases a single application of formula (33) was necessary, excepting as a check on the results by the differential formula. To aid in further calculations, the data of Table 3 have been calculated, and by interpolation from this table it should be possible to derive the proper value of f without recourse to formulas (38) and (39).

EXAMPLE 3.—CALCULATION OF y_m BY FORMULA (37)

To illustrate the use of formula (37) we may take the same case as that in the preceding example, but will start from the final value found in the latter, and show that it is consistent with (37). The mode of procedure is, however, the same, in case only a first approximation is available, except that the factor f is to be calculated from the differential formulas (40) and (42).

Starting from the value $\log y = \bar{1}.5838040$, we find for q_1 the value 0.01131756, to which corresponds the value $X = 0.812335$, which is obtained by interpolation in Table 2. Direct calculation of X from equation (37) gives the value 0.8123349 (see example 7).

The calculation of y_m by formula (37) is then as follows:

$$\begin{aligned}
 \log 32 &= 1.5051500 \\
 " \quad q_1 &= 2.0537527 \\
 " \quad \alpha &= 1.6989700 \\
 " \quad X &= 1.9097351 \\
 2 \log y_m &= 1.1676078 \\
 \log y_m &= 1.5838039 \quad y_m = 0.3835340
 \end{aligned}$$

The value of the logarithm of y_m just found agrees as closely with that found by (33) as the accuracy of the logarithms will allow. The results by the two formulas agree within, at most, one part in four million. This case must be regarded as giving a very searching test of the accuracy of the coefficients in the two formulas, since terms of the eighth order have to be taken into account in each of the formulas.

EXAMPLE 4.—ILLUSTRATION OF THE USE OF CURVE (FIG. 2) FOR OBTAINING A FIRST APPROXIMATION

Let us take the value $\alpha = 0.45$. For this value of α the curve, Fig. 2, gives

$$\begin{aligned}
 \psi &= \frac{\delta}{\alpha} = 46^\circ 13'.6 \text{ or } 46^\circ 13' 36'' \\
 \delta &= 46^\circ 13' 36'' \times 0.45 = 20^\circ 48' 7''.2 \\
 \log \alpha &= \log 0.45 &= 1.6532125 & 1 - \alpha^2 = 1 - 0.2025 \\
 " \quad M &= " \tan \delta &= 1.5796743 & = 0.7975 \\
 " \quad \frac{\alpha}{M} & &= 0.0735382 \\
 2 " \quad " & &= 0.1470764 \\
 " \quad (1 - \alpha^2) & &= 1.9017307 \\
 \therefore " \quad \frac{M^2}{\alpha^2} (1 - \alpha^2) &= 1.7546543 & \frac{M^2}{\alpha^2} (1 - \alpha^2) = 0.5684003 \\
 \log \left[1 - \frac{M^2}{\alpha^2} (1 - \alpha^2) \right] &= 1.6350811 \\
 1 - \frac{M^2}{\alpha^2} (1 - \alpha^2) &= 0.4315997 \\
 \frac{1}{2} " & &= 1.8175406 \\
 \sqrt{1 - \frac{M^2}{\alpha^2} (1 - \alpha^2)} &= 0.6569626
 \end{aligned}$$

$$\log \left[1 - \sqrt{1 - \frac{M^2}{\alpha^2} (1 - \alpha^2)} \right] = \bar{1}.5353416$$

$$1 - \sqrt{1 - \frac{M^2}{\alpha^2} (1 - \alpha^2)} = 0.3430374$$

$$\log \frac{\alpha}{M} = 0.0735382$$

$$\text{sum} = \log \gamma_m = \bar{1}.6088798 = \log \text{ first approximation.}$$

Carrying out the calculation by means of (33) we find $\log q = \bar{2}.9887127$, $\log P = \bar{1}.9271036$ (Table 1) and, as a second approximation $\log \gamma_m = \bar{1}.6088385$, which is a difference in the logarithm of -413 in the seventh place. The factor f for this case is 0.2874 (Table 3), which gives a final difference of -580 . Subtracting this from the logarithm of the first approximation, we have the final value $\log \gamma_m = \bar{1}.6088218$. Using this value of $\log \gamma_m$ in (33) the value found is $\log \gamma_m = \bar{1}.6088217$, which is sensibly the same, showing that the correct value is, in this case, found with only two applications of the main formula, and of these the last is necessary as a check only.

The value of the angle ψ corresponding to the correct value of γ_m is $46^\circ 13' 22''.62$, which is only 0.2 of a minute less than the value taken from the curve. This corresponds to about 1 part in 10 000 difference between the correct value and the first approximation.

EXAMPLE 5.—FORMULA (37), USED IN CONNECTION WITH A FIRST APPROXIMATION TAKEN FROM THE CURVE (FIG. 2)

For $\alpha = 0.65$ the curve (Fig. 2) gives $\psi = 46^\circ 28'.5$, and a calculation exactly like that in the preceding example yields, as a first approximation, $\log \gamma_m = \bar{1}.4750232$. Substituting this value in (37), we find $\log q_1 = \bar{3}.6893759$, or $q_1 = 0.004890755$, to which corresponds $\log X = \bar{1}.9426711$ (Table 2), and the second approximation $\log \gamma_m = \bar{1}.4750552$. The logarithms of the first and second approximations here differ by 320 in the seventh place. For this case Table 3 gives $f = 0.3725$, and the final correction is 1.594 times 320, or 510, which, added to the first approximation, yields the value $\log \gamma_m = \bar{1}.4750742$. A check calculation using this value in (37) gives a value of the logarithm greater by one unit in the seventh place. The true value of ψ is about 0.15 of a minute larger than the value taken from the curve.

EXAMPLE 6.—CALCULATION OF THE FACTOR P IN EQUATION (33a)

In the preceding examples the factor P in equation (33a) has been obtained from Table 1 (see Appendix) by interpolation, using second and third differences. For this interpolation we use the formula

$$P = P_0 + h\Delta_1 + \frac{h(h-1)}{2}\Delta_2 + \frac{h(h-1)(h-2)}{3}\Delta_3 + \dots$$

where P_0 is the value of P corresponding to the nearest tabular value q_0 of q , lower than that for which P is desired, h is the fraction of the tabular interval by which the actual value of q exceeds q_0 , and Δ_1 , Δ_2 , Δ_3 are the first, second, and third differences, respectively, given in the table.

Assuming $q = 0.11016235$, the value in example 1, we find

$$\begin{aligned} P_0 &= 0.808160 & \Delta_1 &= -3074.5 & \Delta_2 &= -14 \\ h\Delta_1 &= -499 & h &= 0.16235 \\ \frac{h(h-1)}{2}\Delta_2 &= +1 \\ P &= \underline{.807662} \end{aligned}$$

In the relatively rare cases where the accuracy of the values of P , derived by interpolation from the Table 1, is not sufficient, we have to calculate the values of the terms of the series in (33) as was done in forming Table 1; retaining enough terms in the series and enough figures in the values of these terms to give the accuracy desired.

Thus for the value of q above, we have $\log q = \bar{1}.0420332$, and the calculation of the terms may be arranged as below. In the first column are given the logarithms of the successive powers of q required in (33). In the second column appear the logarithms of the terms of the series. These are obtained with little trouble if the logarithms of the constant coefficients in the series are written along the edge of a piece of paper, which may be placed under the numbers of the first column in such a way that the logarithm of any required coefficient may be brought directly under the logarithm of the corresponding power of q , and thus the addition of logarithms necessary may be readily carried out. In the third

column are grouped the values of the terms of the series, the positive terms being grouped together first and under them the negative terms. The value of P found shows that its value as interpolated from Table 1 is accurate to the number of places given in the table.

Logarithms of powers of q	Logarithms of terms of series	Terms of series
<u>1.0420332</u>		
2.0840664	1.3393389n	1.0293080
4.1681328	2.4669854	.0004674
6.2521992	3.5579804n	.0000090
8.3362656	4.6696524	.0000002
10.4203320	5.8050832n	1.0297846 = Sum of positive terms.
12.5043984	6.956158	-0.2184434
14.5884648	6.116810n	.0036139
16.6725312	7.283643	.0000638
		.0000013
		0.2221224 = Sum of negative terms.
		$P = 0.8076622$

**EXAMPLE 7.—CALCULATION OF THE FACTOR X IN EQUATIONS (37)
AND (37a)**

To illustrate the use of these formulas let us take the case assumed in example 3, where we found $\log q_1 = 2.0537527$.

The formula requires that the quantity $\log_e \frac{1}{q_1}$ shall be calculated. This may be quickly and accurately obtained by first writing $\log_{10} \frac{1}{q_1}$, and then performing the multiplication by the modulus ($\log_e 10 = 2.3025851$) by the aid of the special table provided for this purpose in collections of six-place and seven-place logarithms. The table referred to contains all the multiples of the modulus up 100 times this quantity. By breaking up the desired logarithm into parts of two figures each the successive partial products may be written down directly from the table, the decimal point being properly placed in each case, and the sum of these products is the result desired.

Thus in the present case

$$\log_{10} q_1 = \bar{2}.0537527$$

$$\log_{10} \frac{1}{q_1} = 1.9462473$$

Tabular value for $1.9 = 4.37491168$

“ “ “ $.046 = 0.10591891$

$.00024 = 0.00055262$

$.0000073 = 0.00001681$

$$\text{Sum} = 4.4814000 = \log_e \frac{1}{q_1}$$

$$\log_{10} \log_e \frac{1}{q_1} = 0.6514137$$

$$\log_{10} \left[12 q_1 \log_e \frac{1}{q_1} \right] = 1.7843476$$

$$\log_{10} \left[120 q_1^2 \log_e \frac{1}{q_1} \right] = 2.8381004$$

The calculation of the series Q , R , S , and T in equation (37) may be carried out by the same method as that used for the series in (33) in the preceding example. The logarithms of the ascending powers of q_1 enter into each of these series. Seven terms are necessary in order to obtain Q to seven places, six terms for R , and only five each for S and T .

Thus we find

$$Q = 0.8609019$$

$$S = 1.2767962$$

$$R = 0.8371244$$

$$T = 1.0257542$$

$$120 q_1^2 R \log_e \frac{1}{q_1} = 0.0576621 \quad 12 q_1 T \log_e \frac{1}{q_1} = 0.6242965$$

$$C_1 = 0.8032398$$

$$C_2 = 0.6524997$$

$$\log C_2 = \bar{1}.8145803$$

$$“ C_1 = \bar{1}.9048452$$

$$“ X = \bar{1}.9097351$$

$$X = 0.8123349$$

The value of X found by interpolation from Table 2, using the interpolation formula given in example 6 is $X = 0.812335$. The amount of labor in a direct calculation of X may be materially reduced if, in the computation of the smaller terms of the series, five-place logarithms are used.

EXAMPLE 8.—USE OF EQUATIONS (38) AND (39)

Table 3, which is based on these equations, will suffice in the usual case, the required values of f being found by interpolation. In cases where it may be necessary to calculate f directly, the terms of the series in (38) may be obtained by the method of example 6.

For the case $\alpha = 0.5$ the terms in (38) are 1.87982, 0.00841, and 0.00020, while the negative terms are 0.04420, and 0.00125, giving the result $f_2 = 0.9215$, for the factor in (38).

For the use of (39) the results of the main calculation by (33) will usually be available. In the present case

$$\begin{array}{rcl}
 \log 10 q^4 & = & \bar{3}.174128 \\
 (1 + 10 q^4) & = & \bar{1}.001493 \\
 \log (1 + 10 q^4) & = & 0.000648 \\
 \log y_m^2 & = & \bar{1}.167608 \\
 \text{" } k^2 & = & \bar{1}.921344 \\
 \text{" } (1 + k') & = & 0.148298 \\
 \text{Sum} & = & \bar{1}.237898 \\
 & & \bar{1}.715439 \\
 \text{Diff.} & = & \bar{1}.522459 = \log f_1, \text{ equation (39)} \\
 & & \bar{1}.964491 = \text{" factor of (38)} \\
 \text{Sum} & = & \bar{1}.486950 = \log f \\
 & & 0.30687 = f
 \end{array}
 \qquad
 \begin{array}{rcl}
 \log k' & = & \bar{1}.609606 \\
 \frac{1}{2} \text{ " } k' & = & \bar{1}.804803 \\
 \frac{3}{2} \text{ " } k' & = & \bar{1}.414409 = \text{Sum} \\
 \text{" } 4\alpha & = & 0.301030 \\
 & & \bar{1}.715439 = \text{Sum}
 \end{array}$$

EXAMPLE 9.—CALCULATION BY FORMULAS (40), (41), AND (42)

Let us consider the same case as in the preceding example, $\alpha = 0.5$.

$$\begin{array}{l}
 \log q_1 = \bar{2}.053753 \\
 \log_e \frac{1}{q_1} = 4.481400 \text{ (see example 7).}
 \end{array}$$

The calculation of the terms in the series for $\frac{dC_1}{dq_1}$ and $\frac{dC_2}{dq_1}$ in (40) offer nothing new, and will be found by the same method as in the preceding examples. The values found are $\frac{dC_1}{dq_1} = -17.2414$, and $\frac{dC_2}{dq_1} = -19.443$.

Making use of the values of C_1 and C_2 found in example 7, the rest of the calculation of the factor f follows readily.

$\frac{1}{C_2} \frac{dC_2}{dq_1} = -29.798$	$\log k^2 = \bar{1}.921344$
	$" \sqrt{k} = \bar{1}.980336$
$\frac{1}{C_1} \frac{dC_1}{dq_1} = -21.465$	
$\text{Diff.} = -8.333$	$" k^{\frac{1}{2}} = \bar{1}.901680 = \text{Sum}$
$\log \text{Diff.} = 0.92080$	$" y_m^2 = \bar{1}.167608$
$" q_1 = \bar{2}.05375$	$" (1+k) = 0.281811$
$\text{Sum} = \bar{2}.97455$	$\text{Sum} = A = \bar{1}.351099$
-0.09431	$\log 4\alpha = 0.301030$
1.00000	$" k'^2 = \bar{1}.219213$
$\text{Sum} = 0.90569$	$\text{Sum} = B = \bar{1}.520243$
$\frac{1}{2} " = 0.45284$	
factor in (40)	$\text{Diff.} = (A - B) = \bar{1}.830856 = \log \text{factor (42)}$
$f = 0.30677$	$\log 0.45284 = \bar{1}.655950$
	$\log f = \bar{1}.486806$

EXAMPLE 10.—CALCULATION OF THE MAXIMUM FORCE BY FORMULAS (34) AND (5)

We have found in examples 2 and 3 for the case $\alpha = 0.5$ the value of $\log y_m = \bar{1}.5838040$, to which corresponds $\log q = \bar{1}.0435322$. This value of q is the quantity which is designated by q_0 in equation (34).

In making the substitution of this value in (34) the method of example 6 gives for the terms in the parenthesis the following values:

1.0244396	
0.0095567	
0.0001089	
0.0000021	
<u>1.0341073</u>	= Sum of positive terms.
-0.0004598	
0.0000138	
<u>0.0000003</u>	
0.0004739	= Sum of negative terms.
1.0336334	= series in parenthesis.

The remainder of the calculation is then

$$\begin{aligned}\log \text{ series} &= 0.0143666 \\ \text{" } \frac{96\pi^2}{\sqrt{5}} &= 2.6270859 \\ \text{" } q_0^2 &= \underline{2.0870644} \\ \text{" } F_m &= 0.7285169 \\ F_m &= 5.352010\end{aligned}$$

If, instead of using (34), we substitute the value of y_m found previously in the general equation for the force, we should, of course, arrive at the same result, since equation (34) was derived by just this process. The value of q will be the same as that in the preceding calculation. The terms in the series are found to be

$$\begin{array}{r} 1.2443963 \\ 0.0335978 \\ 0.0033575 \\ 0.0002702 \\ 0.0000185 \\ 0.0000011 \\ \hline \text{Sum} = 1.2816414 \\ \log \text{ sum} = 0.1077666 \quad \log q^2 = 2.0870644 \\ \text{" } y_m = 1.5838040 \quad \text{" } \sqrt{q} = \underline{1.5217661} \\ \text{" } 192\pi^2 = \underline{3.2776009} \quad \text{" } q^{\frac{5}{2}} = \underline{3.6088305} \\ \quad 2.9691715 \quad \text{" } \sqrt{\alpha} = \underline{1.8494850} \\ \quad \underline{3.7593455} \quad \quad \quad \underline{3.7593455} \\ \log F_m = 0.7285170\end{array}$$

which gives a value of F_m in agreement with the result by the preceding calculation, thus checking (34).

EXAMPLE 11.—CALCULATION OF THE MAXIMUM FORCE, FORMULAS (37) AND (6)

The maximum force will be calculated in this example for the same case as in the preceding example to show the agreement of the formulas. In both cases the higher order terms of the series are important. The value of q_1 , corresponding to the position of the coils for maximum force, has already been derived in example 3. It is $\log q_1 = 2.0537527$.

The two sets of terms in the parentheses in (6) are then as follows:

1.1358107	1.0076852
0.0017854	0.0000213
<u>0.0000040</u>	<u>1.0077065</u>
1.1376006	
- 0.0245927	- 0.1131756
0.0000924	0.0004349
<u>0.0000001</u>	<u>0.0000009</u>
- 0.0246852	- 0.1136114
Series = 1.1129154	0.8940951 = series.
Second term = <u>0.5441659</u>	10.729141 = multiplied by 12.
Parenthesis = 0.5687495	

log =	1.0305649	
$\log_{10} \log_e \frac{1}{q_1}$	0.6514137	
$\log_{10} q_1$	<u>2.0537527</u>	
log second term =	1.7357313	$\log \sqrt{\alpha} = 1.8494853$
log parenthesis =	1.7549210	$q_1 = \frac{2.0537527}{3.9032377}$
$\log \frac{\pi}{16} =$	1.2930299	
" $y_m =$	<u>1.5838040</u>	
	2.6317549	
	<u>3.9032377</u>	
$\log F_m =$	0.7285172	
$F_m =$	5.352014	

This result differs from that by the other independent formula (5) by no more than the errors of calculation with seven-place logarithms.

EXAMPLE 12.—THE COEFFICIENTS η AND ϵ

These coefficients have been calculated for several values of α and are given in Table 5. To show the agreement between the two sets of formulas, in the range where both may be used, the following values may be cited for the case $\alpha = 0.5$.

	By (48) and (51)	By (55) and (58)	B. S. current balance
$\alpha = 0.5$	$\eta = -0.62813$	$\eta = -0.62808$	$\eta = -0.62$
	$\epsilon = 2.5494$	$\epsilon = 2.5493$	$\epsilon = 2.555$

For the case $\alpha=0.4$, the agreement of the two formulas is as good, the mean results being $\eta=-0.35298$ and $\epsilon=2.3102$. The values found experimentally with the B. S. current balance were, for $\alpha=0.4$, $\eta=-0.35$ and $\epsilon=2.317$. The agreement is to be regarded as satisfactory when it is remembered that the values calculated by the formulas are differential coefficients, whereas the experimental values refer to finite displacements.

EXAMPLE 13.—ILLUSTRATING THE METHOD FOR CALCULATING THE MAXIMUM FORCE FOR COILS WHOSE RADII HAVE A RATIO DIFFERING SLIGHTLY FROM ONE OF THE VALUES OF TABLE 4

Consider the case of a pair of coils whose radii were intended to have a ratio of 1:2, but whose actual values have a ratio of $\alpha=0.5(1.01)=0.505$.

Here $\frac{\Delta\alpha}{\alpha_0}=0.01$, if we take $\alpha_0=0.5$. For the value $\alpha_0=0.5$, Table 4 gives $\log F_m=0.7285170$, and Table 5 gives for $\alpha=0.5$, $\epsilon=2.5493$; for $\alpha=0.55$, $\epsilon=2.7205$, with the tabular differences $\Delta_1=0.1712$, $\Delta_2=0.0507$, and $\Delta_3=0.0240$.

To obtain ϵ_1 , the value corresponding to $\alpha=0.505$, we have to interpolate for an interval of 0.1 of that used in the table, and we find the result $\epsilon_1=2.5493+0.0171-0.0023=2.5641$.

This gives, therefore, as a first approximation, $\frac{\Delta F}{F_0}=0.01 \times 2.56$ and, consequently, $\frac{F_1\alpha_0}{F_0\alpha_1}=\frac{1.0256}{1.01}=1.01545$, and $\epsilon_1'=2.5641 \times 1.01545=2.6037$. As a first approximation, therefore, the mean value over the interval is $\frac{1}{2}(2.5493+2.6037)=2.5765$. Using this value, we have, as a second approximation, $\frac{F_1\alpha_0}{F_0\alpha_1}=\frac{1.025765}{1.01}=1.01561$, which gives $\epsilon_1'=2.5641 \times 1.01561=2.6041$, and the second approximation to the mean value is $\epsilon=2.5767$, which is very little different from the first approximation.

We have, therefore, $\frac{\Delta F}{F_0}=0.025767$, and the required value of F_m for $\alpha=0.505$ is found by multiplying the tabular value of F_m for $\alpha_0=0.5$ by the factor 1.025767.

$$\log (1.025767) = 0.0110487$$

$$\text{" } F_0 \quad \quad = 0.7285170$$

$$\log F_m = 0.7395657 \quad F_m = 5.489916 \text{ for } \alpha = 0.505.$$

To check this calculation, we may find the value of η for this same interval. Table 5 gives

$$\eta_0 = -0.62810 \text{ for } \alpha_0 = 0.5$$

$$\eta_1 = -0.64591 \text{ interpolated for } \alpha_1 = 0.505$$

$$\eta_1 \frac{\alpha_0}{\alpha_1} \frac{y_1}{y_0} = -0.64591 \left(\frac{1 - .00636}{1.01} \right) = -0.63545$$

$$\eta_1' = -0.63178$$

A second approximation gives $\eta_1' = -0.63179$. Using the value $\frac{\Delta y}{y_0} = -0.0063179$, we find $\bar{1}.5810514$ as the value of $\log y_m$ derived from the tabular value for $\alpha_0 = 0.5$. Using this as a first approximation, we find by use of the main equation (33) the slightly more accurate value $\log y_m = \bar{1}.5810517$ for $\alpha = 0.505$, and substituting this result in (34) the force is found to have a value of 5.489926, a value only two parts in a million different from the value found by the interpolation process. This difference is not large when we consider the largeness of the interval over which the interpolation was made and the very considerable magnitude of the second and third differences of the tabulated values of ϵ . The amount of work necessary for finding the value of the maximum force by use of the interpolated value of ϵ is, of course, vastly less than that required for the application of the exact formulas.

EXAMPLE 14.—CALCULATION OF THE MAXIMUM FORCE BETWEEN THE BUREAU OF STANDARDS CURRENT BALANCE COILS AND THE DISTANCES BETWEEN THE COILS

To illustrate the use of the formulas of this article for actual coils, the complete calculations are given of the constants of the coils of the Bureau of Standards current balance. The work is arranged for easier following in tabular form (Tables A and B); all the steps, except some of minor importance, are given in the compass of the table. For comparison, the published results given in Tables IX and XI of the article on the determination of

the international ampere are appended. The latter values, as has already been explained, were obtained by interpolation between the values of the force calculated by formula (4) for a number of distances of the coils in the neighborhood of the position corresponding to maximum force.

TABLE A

Calculation of Maximum Distance z_m for Bureau of Standards Coils

Coils	$\frac{\Delta\alpha}{d}$	Interpolated value η_1	$\frac{\eta_1'}{\eta_1} \frac{\alpha_0}{\alpha_1} \frac{y_m}{y_0}$	η mean	$\frac{\Delta y}{y_0}$	$\log y_m$	z_m	Values current balance
M_2L_1	-0.0010800	-0.62620	-0.62715	-0.62755	+0.0006778	1.5840983	9.60946	9.6092
M_2L_2	-.0010566	.62625	.62733	.62772	6632	.5840919	9.60910	9.6089
M_2L_3	+.0002793	.62859	.62831	.62822	-.0001755	.5837278	9.58822	9.5882
M_2L_4	.0000908	.62826	.62817	.62813	- 570	.5837792	9.59117	9.5912
M_3S_1	+.0047609	-0.63652	-0.63158	-0.62984	-.0029986	1.5824998	7.64009	7.6400
M_3S_2	.0052139	.63731	.63190	.63000	32848	.5823750	7.63445	7.6344
M_3L_1	+.0022365	-0.35492	-0.35384	-0.35341	-.0007904	1.6294549	10.66733	10.6673
M_3L_2	.0022600	.35503	.35394	.35346	7981	.6294515	10.66700	10.6669
M_3L_3	.0036000	.35611	.35438	.35368	-.0012732	.6292449	10.64769	10.6476
M_3L_4	.0034113	.35595	.35431	.35364	12064	.6292741	10.65041	10.6502
M_4L_1	-.0038325	-0.62133	-0.62520	-0.62665	+0.0024016	1.5848458	9.62602	9.6256
M_4L_2	-.0038091	.62127	.62512	.62661	23868	.5848394	9.62565	9.6253
M_4L_3	-.0024768	.62365	.62617	.62714	15533	.5844781	9.60480	9.6047
M_4L_4	-.0026649	.62341	.62611	.62711	16712	.5845292	9.60775	9.6077

It must be remembered that the maximum force here calculated is not quite equal to the maximum force between the actual coils, but is the value between the equivalent filaments by which the coils may be replaced, on the assumption that their cross sections are square. It is the force F_1 in the article referred to, and the correction for the actual deviation from a square cross section must be applied as there shown (pp. 329 to 334).

The equivalent radii of the coils, derived by Lyle's Method, are given there in Table IX. They are:

Coil	Equivalent radius	Coil	Equivalent radius
M_2	12.50552	S_2	19.97113
M_3	10.03763	L_1	25.03808
M_4	12.47106	L_2	25.03749
S_1	19.98014	L_3	25.00405
		L_4	25.00877

Table A covers the calculation of the distance z_m for maximum force. Column 2 gives the fractional differences between the actual ratios of the radii and the nominal values α_0 . The latter is $\alpha_0=0.4$ for the movable coil M_3 , when used with the fixed coils L_1, L_2, L_3 , and L_4 ; for the other combinations it is 0.5. Column 3 gives the interpolated values of η_1 , and column 4 the same quantity multiplied by the correction factor $\frac{\alpha_0}{\alpha_1} \frac{y_1}{y_0}$.

In columns 5 and 6 appears the mean value η and the change in $y_0, \frac{\Delta y}{y_0} = \eta \frac{\Delta \alpha}{\alpha_0}$ for the actual fractional deviations of the coils from the nominal ratio of the radii. The values of column 7 are found by adding the logarithms of $\left(1 + \frac{\Delta y}{y_0}\right)$ to the values of $\log y_0$ for the nominal ratio taken from Table 4, viz:

For $\alpha_0=0.5 \log y_0=\overline{1.5838040}$

“ $\alpha_0=0.4$ “ $y_0=\overline{1.6297983}$

TABLE B

Calculation of the Maximum Force for the Bureau of Standards Coils

Coils	$\frac{\Delta \alpha}{\alpha_0}$	Inter- polated value ϵ_1	$\frac{\epsilon_1'}{\alpha_0 F_m}$ $=\epsilon_1 \alpha_1 F_0$	ϵ mean	$\frac{\Delta F}{F_0}$	$\log \left(1 + \frac{\Delta F}{F_0}\right)$	$\log F_1$	F_1	F_1 values current balance
M_2L_3	+0.0002793	2.5493	2.5510	2.5502	+0.0007123	0.0003092	0.7288262	5.355823	5.355834
M_2L_4	+ .0000908	2.5493	2.5499	2.5496	.0002315	.0001006	0.7286176	5.353251	5.353251
M_3S_1	+0.0047609	2.5563	2.5754	2.5623	+0.0121987	0.0052658	0.7337828	5.417298	5.417298
M_3S_2	.0052139	2.5569	2.5776	2.5635	.0133658	.0057662	0.7342832	5.423544	5.423552
M_3L_1	+0.0022365	2.3118	2.3186	2.3144	+0.0051761	0.0022422	0.4966225	3.137780	3.137781
M_3L_2	.0022600	2.3119	2.3187	2.3145	.0052308	.0022658	.4966461	3.137951	3.137953
M_3L_3	.0036000	2.3130	2.3239	2.3171	.0083416	.0036077	.4979880	3.147662	3.147665
M_3L_4	.0034113	2.3129	2.3232	2.3167	.0079030	.0034187	.4977990	3.146292	3.146292
M_4L_2	−0.0024768	2.5458	2.5360	2.5426	−0.0062974	$\overline{1.9972564}$	0.7257734	5.318308	5.318310
M_4L_4	− .0026649	2.5454	2.5349	2.5421	− .0067744	.9970479	.7255649	5.315754	5.315754

The values of the distance z_m then follow on multiplying the values in column 7 by the radius A of the larger coil. The values published in Table IX appear in the last column. The differences between the latter and the values here calculated are very small and are not systematic.

In Table B appear the calculations of the maximum force. These should be clear from the description of the preceding table. The values of the maximum force F_0 , corresponding to the nominal ratios 0.5 and 0.4, are taken from Table 4, and form the basis of the calculations. In the last column are given the values of F_m from Table XI of the article on the current balance. The differences between these values and those here calculated are, on the average, less than a part in a million. In fact, the agreement is better than for the distances z_m in the preceding table. This is explained when it is remembered that the force changes very slowly in the neighborhood of the critical distance, and thus makes it difficult to obtain the actual position of the coil for maximum force by interpolation.

X. SUMMARY

1. The force of attraction or repulsion between two parallel, coaxial circular currents has its maximum value for a distance z_m , between their planes, such that the ratio $\frac{z_m}{A}$ is a function of the ratio of the radii $\frac{a}{A}$ alone.

2. The value of the force between the two circular filaments, when one ampere flows in each, is a function of the ratio of the radii alone. A knowledge of the maximum value of this force is of practical importance in the absolute measurement of current, since the Rayleigh current balance employs two coils placed experimentally at the distance which gives the maximum force between them. The calculation of the constant of this current balance postulates a knowledge of the maximum value of the force in terms of the dimensions of the coils.

4. Formulas have been developed by Maxwell and Nagaoka which allow the force to be calculated between any two parallel, coaxial currents whose planes are at any desired distance z apart. Previously the maximum value of the force has been derived by interpolation, graphically or otherwise, between values of the force calculated for various positions of the coils in the region where the force is known to be a maximum.

5. In this paper, by a method which is an extension of that used by Nagaoka in deriving his formulas for the force between the coils, two equations are derived, which give the distance between the coils corresponding to maximum force, as a function of the ratio of the radii, α . These relations are in the form of q series, one depending on a modulus which is complementary to that of the other. One or the other of these formulas will be found to converge well for any desired value of α .

6. Unfortunately, these relations do not, apparently, allow of being put in a form such as to allow the distance z_m to be calculated directly. They may, however, be solved for z_m by successive approximation without serious difficulty.

7. Starting from Rayleigh's approximation formula for z_m , which gives precise values over only a limited range of values of the ratio of the radii, methods are here worked out for facilitating the process of approximation, together with an empirical method for obtaining a more accurate first approximation than can, in general, be obtained by the formula of Rayleigh.

8. Tables are given of the distance z_m and the maximum force F_m for values of the ratio of the radii advancing in steps of 0.05, as well as a number of tables of auxiliary quantities, which materially reduce the labor of calculation of the distance z_m and the maximum force for such values of α as are not included in the tables.

9. A method is also given for easily obtaining the values of the desired quantities from the fundamental values in the tables for coils having radii whose ratio differs only slightly from one of the nominal values included in the tables. This covers the case of current balance coils in practice.

10. The use of the formulas and tables is thoroughly illustrated by examples so designed as to show the agreement of the different methods. Finally, a comparison is given of the results of calculations made for the coils of the Bureau of Standards current balance, by the method of this paper, with the published values obtained by the interpolation process. The agreement is, on the average, closer than one part in a million. For practical cases like this the use of the tables given here enables the constants to

be found with vastly less labor than in a general case not covered by the fundamental values in the tables. Even in the general case, however, the method of this paper is likely to be less time consuming than the interpolation method, and has the advantage of furnishing a check on the latter.

WASHINGTON, January 1, 1915.

APPENDIX

TABLE 1

Table of Values of the Quantity in Brackets in Equation 33 (Factor P in Equation (33a))

q	P	d_1	d_2	q	P	d_1	d_2
0	1.000000	- 18	-36	0.040	0.971701	-1406	-32.5
0.001	0.999982	54	36	.041	.970295	1438.5	32
.002	.999928	90	36	.042	.968856	1470.5	31.5
.003	.999838	126	36	.043	.967386	1502	32
.004	.999712	162	36	.044	.9658835	1534	31
.005	.999550	198	36	.045	.964350	1565	31
0.006	0.999352	- 234	-36	0.046	0.962784	-1596	-31.5
.007	.9991185	270	35.5	.047	.961188	1627.5	31
.008	.998849	305.5	35.5	.048	.959560	1658	31
.009	.998543	341	36	.049	.957902	1689	30
.010	.998202	377	36	.050	.956213	1719	30
0.011	0.997825	- 413	-35	0.051	0.954494	-1749.5	-30
.012	.997412	448	36	.052	.952744	1779.5	30
.013	.996964	484	36	.053	.950965	1809	29.5
.014	.996480	520	35	.054	.9491555	1839	29
.015	.995960	555	35	.055	.947317	1868	29
0.016	0.995405	- 590	-36	0.056	0.945449	-1897	-29
.017	.994815	626	35	.057	.943552	1926	29
.018	.994189	661	35	.058	.941626	1954.5	28.5
.019	.993528	696	35	.059	.939671	1983	28
.020	.992832	731	35	.060	.937688	2011	28
0.021	0.992101	- 766	-35	0.061	0.935677	-2039	-28
.022	.991334	801	35	.062	.933638	2067	27
.023	.990533	836	34	.063	.9315715	2094	27
.024	.989698	870	35	.064	.929478	2121	27
.025	.988827	905	34	.065	.927356	2148	27
0.026	0.987922	- 939	-35	0.066	0.925208	-2175	-26
.027	.986983	974	34	.067	.9230335	2201	26
.028	.986009	1007	34	.068	.920832	2227	26
.029	.985002	1042	34	.069	.918605	2253	26
.030	.983960	1076	-33.5	.070	.916352	2279	25
0.031	0.982884	-1109.5	-33.5	0.071	0.914073	-2304	-25
.032	.9817745	1143	34	.072	.911769	2329	25
.033	.980631	1177	33	.073	.909440	2354	24
.034	.979455	1210	33	.074	.9070855	2379	24
.035	.978245	1243	33	.075	.904707	2403	24
0.036	0.977002	-1276	-33	0.076	0.902304	-2427	-24
.037	.975726	1309	32.5	.077	.899877	2451	23.5
.038	.974417	1341.5	32.5	.078	.897426	2474	23
.039	.973075	1374	32	.079	.894952	2497	23
.040	.971701	1406	32.5	.080	.892455	2520	23

TABLE 1—Continued

Table of Values of the Quantity in Brackets in Equation 33 (Factor P in Equation (33a))—Continued

<i>q</i>	<i>P</i>	<i>A</i> ¹	<i>A</i> ²	<i>q</i>	<i>P</i>	<i>A</i> ¹	<i>A</i> ²
0.080	0.892455	—2520	—23	0.100	0.838072	—2919	—17
.081	.889934	2543	22.5	.101	.835153	2935	17
.082	.8873915	2565	22	.102	.8322175	2952	16
.083	.884826	2587	22	.103	.829265	2968	16
.084	.882239	2609	22	.104	.826297	2985	15
.085	.879630	2631	21.5	.105	.823312	3000	15
0.086	0.876999	—2652	—21	0.106	0.820312	—3015.5	—15
.087	.874347	2673	21	.107	.8172965	3031	15
.088	.871674	2694	20.5	.108	.814266	3046	14
.089	.868981	2713.5	20	.109	.811220	3060	14
.090	.866267	2734	19.5	.110	.808160	3074.5	14
0.091	0.863533	—2754	—19	0.111	0.805086	—3088	—14
.092	.860779	2773	19.5	.112	.801998	3102	13.5
.093	.858006	2793	19	.113	.798896	3115	13
.094	.855213	2811	19	.114	.795781	3128	13
.095	.852402	2830	18	.115	.792652	3141	13
0.096	0.849572	—2848	—18	0.116	0.789511	—3154	—12
.097	.846724	2867	17.5	.117	.786358	3166	12
.098	.843857	2884	17.5	.118	.783192	3178	12
.099	.840973	2901.5	17	.119	.780014	3190	12
.100	.838072	2919	17	.120	.776825	3202	12

TABLE 2.

Table of Values of Factor X in Equation (37'a.)

<i>q</i> ₁	<i>X</i>	<i>A</i> ₁	<i>A</i> ₂	<i>q</i> ₁	<i>X</i>	<i>A</i> ₁	<i>A</i> ₂
0	1.000000	—7058	+1654	0.0015	0.942127	—2667	+69.5
0.0001	0.992942	5404	620	.0016	.939461	2597	65.5
.0002	.9875375	4784	400	.0017	.9368635	2532	62
.0003	.982753	4384	296	.0018	.934332	2470	58
.0004	.978369	4088	235	.0019	.931862	2412	54
.0005	.974281	3853	195	.0020	.929450	2357	52
0.0006	0.970428	—3658	+166	0.0021	0.927093	—2305	+50
.0007	.966770	3493	144	.0022	.9247875	2256	47
.0008	.963277	3348	128	.0023	.922532	2208	45
.0009	.959929	3221	114	.0024	.920324	2163	43
.0010	.956708	3107	104	.0025	.918160	2120	41
0.0011	0.953601	—3002.5	+94	0.0026	0.916040	—2079	+39
.0012	.950599	2909	87	.0027	.913962	2039	38
.0013	.947690	2822	80.5	.0028	.911922	2001	37
.0014	.944869	2741	74.5	.0029	.909921	1964	35
.0015	.942127	2667	69.5	.0030	.907957	1929	34

TABLE 2—Continued

Table of Values of Factor X in Equation (37'a.)—Continued

q_1	X	d_1	d_2	q_1	X	d_1	d_2
0.0030	0.907957	-1929	+34	0.0075	0.844435	-1029	+11
.0031	.906028	1895	33	.0076	.843406	1018	12
.0032	.904133	1862	32	.0077	.842388	1006	12
.0033	.902271	1830.5	31	.0078	.8413825	994	11
.0034	.900440	1800	30	.0079	.8403885	983	12
.0035	.898640	1770	29	.0080	.839406	971	11
0.0036	0.896870	-1741	+28	0.0081	0.838435	-960	+11
.0037	.895129	1713	27	.0082	.837474	949	11
.0038	.893416	1686	26	.0083	.836525	938	10
.0039	.891730	1660	25	.0084	.835587	928	10.5
.0040	.890070	1634	25	.0085	.834659	917.5	10.5
0.0041	0.888436	-1610	+25	0.0086	0.833742	-907	+10
.0042	.886826	1585	24	.0087	.832835	897	10
.0043	.885241	1561.5	23	.0088	.831938	887	10
.0044	.883679	1539	22	.0089	.831051	877	10.5
.0045	.882141	1516.5	22	.0090	.830174	867.5	9.5
0.0046	0.880624	-1495	+21	0.0091	0.829307	-858	+10
.0047	.8791295	1473.5	21	.0092	.828449	848	9
.0048	.877656	1453	21	.0093	.827601	839	10
.0049	.876203	1432	19	.0094	.826763	829	8
.0050	.874771	1413	19.5	.0095	.825933	821	10
0.0051	0.873359	-1393	+19	0.0096	0.825113	-811	+8
.0052	.871965	1374	18	.0097	.8243015	803	10
.0053	.870591	1356	18	.0098	.823499	793	8
.0054	.869235	1338	17.5	.0099	.822705	785	9
.0055	.867898	1320	17	.0100	.821920	776	8
0.0056	0.8665775	-1303	+17	0.0101	0.821144	-768	+8
.0057	.865275	1286	16.5	.0102	.820375	760	8
.0058	.863989	1269	16	.0103	.819616	752	9
.0059	.862720	1253	16	.0104	.818864	743	7
.0060	.861467	1236.5	15.5	.0105	.818121	736	8
0.0061	0.8602305	-1221	+15	0.0106	0.817385	-728	+8
.0062	.859009	1206	15	.0107	.816657	720	8
.0063	.857803	1191	15	.0108	.815938	712	7
.0064	.856613	1176	14.5	.0109	.815226	705	9
.0065	.8554365	1161	14	.0110	.814521	696	7
0.0066	0.8542755	-1147	+13.5	0.0111	0.813825	-689	+7
.0067	.853128	1133.5	13	.0112	.813136	682	7
.0068	.851995	1119	13	.0113	.812454	675	8
.0069	.850876	1106	13	.0114	.811779	667	7
.0070	.849770	1093	13	.0115	.811112	660	6.5
0.0071	0.848677	-1080	+13	0.0116	0.810452	-653.5	+7.5
.0072	.847597	1066	13	.0117	.8097985	646	7
.0073	.846531	1054	12	.0118	.809153	639	6.5
.0074	.845477	1042	13	.0119	.8085135	632.5	6.5
.0075	.844435	1029	11	.0120	.807881	626	6

TABLE 3
Values of Factors in Differential Equations (38) to (42)

$\alpha = \frac{a}{A}$	logarithms of factors		$\log f$	f	$\log F = \log \frac{F}{1-f}$	F	A_1	A_2
	in (38)	in (39)						
0	1.69897n	1.60206n	1.30103	0.20000	0.09691	1.2500	+ 18	+ 34
0.05	.70209	.60136	.30345	.20112	.09752	.2518	52	37
0.10	.71137	.59926	.31063	.20447	.09934	.2570	89	36
0.15	.72657	.59573	.32231	.21004	.10240	.2659	125	39
0.20	.74735	.59072	.33807	.21780	.10668	.2784	164	41
0.25	.77328	.58406	.35734	.22769	.11221	.2948	205	42
0.30	1.80383	1.57587	1.37969	0.23971	0.11902	1.3153	+247	+ 46
0.35	.83860	.56579	.40438	.25374	.12711	.3400	293	48
0.40	.87716	.55370	.43086	.26969	.13649	.3693	341	52
0.45	.91919	.53936	.45855	.28744	.14718	.4034	393	55
0.50	1.96448	.52246	.48695	.30686	.15918	.4427	448	58
0.55	0.01292	1.50259	.51551	.32773	0.17246	1.4875	506	

α	logarithms of factors		$\log f$	f	$\log F$	F	A_1	A_2
	in 40	in 42						
0.45	1.65906	1.79941	1.45847	0.28739	0.14715	1.4033	+392	+ 55
0.50	.65595	.83086	.48681	.30677	.15912	.4425	447	56
0.55	1.65597	1.85936	.51534	0.32759	0.17237	1.4872	+503	+ 58
0.60	.65834	.88523	.54357	.34960	.18682	.5375	561	55
0.65	.66246	.90864	.57110	.37248	.20237	.5936	616	47
0.70	.66782	.92970	.59755	.39584	.21885	.6552	663	30
0.75	.67391	.94841	.62232	.41910	.23590	.7215	693	+ 1
0.80	1.68030	1.96471	1.64501	0.44159	0.25305	1.7908	694	- 50
0.85	.68659	.97845	.66504	.46241	.26955	.8602	644	-132
0.90	.69229	.98932	.68161	.48041	.28434	.9246	512	-270
0.95	.69678	1.99684	.69362	.49388	.29575	1.9758	242	
1.00	1.69897	0.00000	1.69897	0.50000	0.30103	2.0000		

TABLE 4

Fundamental Values of the Maximum Force Ratio $\gamma_m = \frac{z_m}{A}$ and the Maximum Force F_m

$\alpha = \frac{a}{A}$	$\log \gamma_m$	γ_m	ψ	$\log F_m$	F_m
0	1.6989700	0.5000000	° ' "		0
0.05	.6979913	.4988745	45 50 31.04	2.6279557	0.04245763
0.10	.6950381	.4954936	45 51 28.52	1.2326395	0.1708596
0.15	.6900572	.4898434	45 53 3.47	1.5892443	0.3883688
0.20	.6829567	.4818998	45 55 14.40	1.8454176	0.7005153
0.25	.6736010	.4716296	45 57 59.40	0.0475214	1.1156332
0.30	1.6618033	0.4589901	46 01 15.60	0.2163112	1.6455504
0.35	.6473125	.4439279	46 04 58.77	0.3629849	2.306667
0.40	.6297983	.4263815	46 09 3.72	0.4943803	3.121622
0.45	.6088217	.4062765	46 13 22.62	0.6150973	4.121898
0.50	.5838040	.3835341	46 17 45.84	0.7285170	5.352011
0.55	1.5539627	0.3580657	46 21 59.60	0.8373702	6.876544
0.60	.5182254	.3297808	46 25 45.82	0.9441070	8.792392
0.65	.4750743	.2985893	46 28 39.62	1.0512140	11.25159
0.70	.4222883	.2644163	46 30 8.59	1.1615723	14.50682
0.75	.3564223	.2272073	46 29 26.32	1.2790169	19.01152
0.80	1.2717509	0.1869610	46 25 30.92	1.4094251	25.66995
0.85	1.1576668	.1437695	46 17 1.94	1.5633218	36.58658
0.90	2.9905951	.0978577	46 02 2.59	1.7638911	58.06188
0.95	2.6960993	.0496706	45 37 59.40	2.0847542	121.54978
1.00		0	45 0 0.00		

TABLE 5

Coefficients η and ϵ in Equations (48), (51), (55), and (58)

$\alpha = \frac{a}{A}$	η	d_1	d_2	d_3	ϵ	d_1	d_2	d_3
0	0	— .00451	— .00919	— .00050	2.0000	+ .0040	+ .0082	+ .0003
0.05	— 0.00451	1370	969	87	.0040	122	85	8
0.10	.01821	2339	1056	129	.0162	207	93	12
0.15	.04160	3395	1185	193	.0369	300	105	14
0.20	.07555	4580	1378	263	.0669	405	119	22
0.25	.12135	5958	.01641	366	2.1074	524	141	33
0.30	— 0.18093	— .07599	— .02007	— .00516	2.1598	+ .0665	+ .0174	+ .0043
0.35	.25692	.09606	2523	731	.2263	839	217	62
0.40	.35298	.12129	3254	1083	.3102	.1056	279	98
0.45	.47427	.15383	4337	— .01573	.4158	.1335	377	130
0.50	.62810	.19720	— .0591	— .0262	2.5493	.1712	507	237
0.55	— 0.82530	— .2563	— .0853	— .0395	2.7205	+ .2219	+ .0744	+ .0380
0.60	1.0816	.3416	.1248	.0763	2.9424	.2963	.1124	690
0.65	1.4232	.4664	.2011	.1466	3.2387	.4087	.1814	.1206
0.70	1.8896	.6675	.3477	.3391	3.6474	.5901	.3020	.3790
0.75	2.5571	1.0152	.6868	—1.0101	4.2375	.8921	.6810	.9679
0.80	— 3.5723	—1.7020	—1.6969	—5.0143	5.1496	+1.5731	+1.6489	+4.9548
0.85	5.2743	3.3989	6.7112		6.7127	3.2220	6.6037	
0.90	8.6722	10.1101			9.9347	9.8257		
0.95	—18.7823				19.7604			



